In Regular Protomodular Categories with an Initial Object, Noetherian Objects are Hopfian and Closed Under Subobjects, Quotients and Extensions

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Abstract.

Classical Noetherian objects in algebra, such as modules over a ring, are characterized by the ascending chain condition on subobjects and exhibit key properties including being Hopfian and closed under submodules, quotients, and extensions. This work presents a categorical generalization of these concepts. Firstly, we extend the notion of Noetherian objects to the setting of regular protomodular categories (see [1]) with an initial object. Within this context, we demonstrate that the core properties are preserved: Noetherian objects, defined via stabilizing chains of subobjects, are Hopfian; any regular epimorphism $x \to x$ is an isomorphism for a Noetherian object x. Here assumptions can be relaxed to local ones. We further demonstrate that the class of Noetherian objects is closed under subobjects, regular quotients, extensions, and in the pointed case under finite products.

Secondly, we show that these results established for regular protomodular categories can be further generalized by abstracting the essential mechanisms into the language of stable weak factorization systems (E, M). We define (E, M)-Noetherian objects based on the stabilization of sequences of M-subobjects. The central finding is that the Hopfian property, all morphisms $x \to x \in E$ are isomorphisms (with assumptions on the initial object), and the closure properties hold even in this broader setting, provided the category has pullbacks and the system (E, M) possesses suitable characteristics, such as stability, weak images (weakening of the lifting condition) and a variation of the short-five-lemma, which mimic the behavior characterizing protomodular setting. This factorization system perspective isolates the key structural ingredients required for Noetherian-like behavior. The framework unifies classical results and applies widely, particularly within regular protomodular categories including loops, groups, rings, R-modules, Lie-algebras, crossed modules, Heyting algebras and hoops.

References

[1] F. Borceux and D. Bourn, *Mal'cev, Protomodular, Homological and Semi-Abelian Categories*, Mathematics and Its Applications, vol. 566, Springer, 2004.