

# Enrichment and families over virtual double categories

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## Abstract.

Enriched category theory gives rise to a 2-functor  $\text{Enr}$  from a suitable 2-category of enrichment bases to the 2-category  $2\text{-CAT}$  of 2-categories, sending each base  $\mathcal{V}$  to the 2-category  $\text{Enr}(\mathcal{V}) = \mathcal{V}\text{-Cat}$  of all (small)  $\mathcal{V}$ -categories. Classically, monoidal categories are taken as the enrichment bases, but there are several extensions taking e.g. bicategories, pseudo double categories, multicategories, and virtual double categories [2] as enrichment bases. In this talk, we will show that the 2-functor  $\text{Enr}$  becomes a *parametric right 2-adjoint* if we take the virtual double categories as the enrichment bases. That is, letting  $\mathbf{VDBL}$  be the 2-category of virtual double categories, we show that the 2-functor  $\text{Enr}_1: \mathbf{VDBL} \rightarrow 2\text{-CAT}/\text{Enr}(1)$ , induced from the 2-functor  $\text{Enr}: \mathbf{VDBL} \rightarrow 2\text{-CAT}$  and the terminal object 1 in  $\mathbf{VDBL}$ , is a right 2-adjoint.

In more detail, we first note that the 2-functor  $\text{Enr}: \mathbf{VDBL} \rightarrow 2\text{-CAT}$  can be decomposed as

$$\mathbf{VDBL} \xrightarrow{\text{Mat}} \mathbf{VDBL} \xrightarrow{\text{Mod}} \mathbf{VDBL}_n \xrightarrow{\mathbf{V}} 2\text{-CAT}, \text{ where}$$

- $\mathbf{VDBL}_n$  is the 2-category of virtual double categories with chosen horizontal units and virtual double functors preserving the chosen horizontal units on the nose,
- $\text{Mat}$  maps each  $\mathbb{D} \in \mathbf{VDBL}$  to the virtual double category  $\text{Mat}(\mathbb{D})$  of *matrices* in  $\mathbb{D}$ ,
- $\text{Mod}$  maps each  $\mathbb{D} \in \mathbf{VDBL}$  to the virtual double category  $\text{Mod}(\mathbb{D})$  of *horizontal monads* in  $\mathbb{D}$  (which is naturally equipped with chosen horizontal units) [3, 1], and
- $\mathbf{V}$  maps each  $\mathbb{D} \in \mathbf{VDBL}_n$  to its *vertical 2-category*  $\mathbf{V}(\mathbb{D})$  [1].

We then observe that, on the one hand, both  $\text{Mod}$  and  $\mathbf{V}$  are right 2-adjoints, and on the other hand,  $\text{Mat}$  is the *polynomial 2-functor*

$$\mathbf{VDBL} \xrightarrow{T^*} \mathbf{VDBL}/(\text{Set}_*)_{\text{hc}} \xrightarrow{\Pi_P} \mathbf{VDBL}/\text{Set}_{\text{hc}} \xrightarrow{\Sigma_S} \mathbf{VDBL}$$

induced by a suitable polynomial

$$1 \xleftarrow{T} (\text{Set}_*)_{\text{hc}} \xrightarrow{P} \text{Set}_{\text{hc}} \xrightarrow{S} 1$$

in  $\mathbf{VDBL}$ , and hence is a parametric right 2-adjoint. A closely related polynomial in  $\mathbf{VDBL}$  induces the 2-functor  $\mathbb{F}\mathbf{am}$  for the family construction for virtual double categories.

## References

- [1] G. S. H. Cruttwell and M. A. Shulman, *A unified framework for generalized multicategories*, Theory Appl. Categ. 24 (2010), 580–655.
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- [3] T. Leinster, *Higher Operads, Higher Categories*, Lond. Math. Soc. Lect. Note Ser., vol. 298, Cambridge University Press, 2004.