Categorical-algebraic characterisations of Lie algebras

X. García Martínez

Xabier García Martínez (xabier.garcia@usc.gal)
Universidade de Santiago de Compostela

Abstract.

Let $\mathfrak{M}$ be a non-abelian variety of non-associative algebras over a field of characteristic zero. In [2] we proved that $\mathfrak{M}$ is *locally algebraically cartesian closed* if, and only if it is the variety of Lie algebras. In this way, we provided a characterisation of Lie algebras amongst all varieties of non-associative algebras in a purely categorical way, using *algebraic exponents*.

The key ideas behind this result involve studying the preservation of coproducts for every object B, of the endofunctor $B\flat(-)\mathfrak{M} \to \mathfrak{M}$, that sends an object X to the kernel of the canonical split extension

$$B\flat X \longrightarrow B + X \underset{i_B}{\overset{\langle 1,0\rangle}{\rightleftarrows}} B$$

Inspired by this result we found another characterisation of Lie algebras in [1]:

Theorem 1. *Suppose that $\mathfrak{M}$ is a non-trivial variety of non-associative algebras over a field of characteristic zero satisfying the following two conditions:*

- *it is Nielsen-Schreier;*
- *for every normal object I, I^2 is also normal.*

Then $\mathfrak{M}$ is the variety of Lie algebras.

In this talk we will begin with a conceptual overview of this result, together with its motivation and its origins. We will relate it with the previous characterisation that used algebraic exponents, linking the injectivity and surjectivity of the comparison map

$$B\flat X + B\flat Y \longrightarrow B\flat(X + Y)$$

with the properties of the theorem. Finally, we will discuss the proof techniques, which apart from the categorical-algebraic methods, they also involve homological and computational algebra, together with Gröbner bases for operads.

Joint work with Vladimir Dotsenko (Université de Strasbourg)

References

[1] V. Dotsenko and X. García-Martínez. *A characterisation of Lie algebras using ideals and subalgebras*, Bull. Lond. Math. Soc. 56 (2024), no. 7, 2408–2423.

[2] X. García-Martínez and T. Van der Linden. *A characterisation of Lie algebras via algebraic exponentiation*, Adv. Math. 341, (2019), 92–117.