# Pretorsion Theories on $(\infty, 1)-$Categories

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## 1 Abstract

In the 1960s, Spencer Dickson, in [3], axiomatizing properties of the category of abelian groups, presented a notion of *torsion theory* on abelian categories, which was soon generalized beyond the abelian setting (see, among many others, [1], [2]), and even, recently, in [4] and particularly [5], to general (and not necessarily pointed) $1-$categories. Classically, a *pretorsion theory* on a category $\mathbf{C}$ is a pair of full replete subcategories $(\mathbf{T}, \mathbf{F})$ such that every morphism between them factors through their intersection, $\mathbf{Z} := \mathbf{T} \cap \mathbf{F}$, and that there is a notion of short exact sequence consisting of a $\mathbf{Z}-$kernel and a $\mathbf{Z}-$cokernel that one may associate to every object in $\mathbf{C}$. A *torsion theory* from this perspective is a pretorsion theory where $\mathbf{Z} = \varnothing$. Pretorsion theories satisfy multitudinous properties, including that $\mathbf{T}$, $\mathbf{F}$ and $\mathbf{Z}$ are closed under certain extensions, that $\mathbf{Z}-$kernels and -co-kernels are respectively monomorphisms and epimorphisms, and that $\mathbf{T}$ and $\mathbf{F}$ are respectively coreflective and reflective subcategories of $\mathbf{C}$.

Here we will propose a notion of pretorsion theory for $(\infty, 1)-$categories, compatible with the aforementioned classical one under the taking of the homotopy category $h\mathscr{C}$ of the $(\infty, 1)-$category $\mathscr{C}$ upon which the pretorsion theory is situated. We shall then show that $(\infty, 1)-$categorical pretorsion theories satisfy some, but not all, of the properties fulfilled by their classical counterparts.

There are a notion of normal torsion theory for stable $(\infty, 1)-$categories [6], as well as a version of two-dimensional torsion theory [8] available in the literature, and we will discuss the the compatibility and relationship of our construction with these two by looking in particular at $(\infty, 1)-$torsion theories in our framework as well as its behavior under truncation.

## References


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