Actions of partial groups, and the higher Segal conditions

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Abstract.

Partial groups were introduced in [1] and played a key role in Chermak's proof of the existence and uniqueness of centric linking systems for saturated fusion systems, a major recent result in p-local finite group theory. A partial group consists of a (suitably compatible) collection of partially-defined n-ary multiplications on a set; this structure can concisely be described as a symmetric simplicial set for which the Segal maps are monomorphisms and which has a single vertex [5]. The most important class of partial groups are the *localities*, each of which comes equipped with an 'action' by conjugation on a certain subcollection of its set of p-subgroups.

Heretofore, the notion of action of partial groups has proved elusive. We'll propose a simple general definition in the symmetric sets framework, inspired by the fibrational perspective in category theory. This will be suitable for capturing a number of expected examples, including the action of a locality on its special collection of subgroups, and more generally for partial groups arising from a partial action of a group [3]. These are examples of *characteristic actions*, where the action itself controls which multiplications are valid in the partial group.

Our main application of characteristic actions is to a connection between partial groups and the d-Segal spaces of Dyckerhoff–Kapranov [2]. The higher Segal conditions are a series of exactness conditions for simplicial spaces coming from certain triangulations of cyclic polytopes. These generalize Segal spaces, and have applications (for $d = 2$, when they are also called *decomposition spaces* [4]) in representation theory, K-theory, geometry, combinatorics, and elsewhere, and are closely connected to ∞-operads and to categories with multivalued composition. For a given partial group, one naturally might wonder if it is d-Segal for some d. The *degree* of a partial group is defined to be the least integer k for which it is $(2k-1)$-Segal, so that the degree 1 partial groups are precisely the groups. We'll discuss why characteristic actions are a key ingredient in degree computations.

References

[1] A. Chermak, *Fusion systems and localities*, Acta Math. 211 (2013), no. 1, 47–139.

[2] T. Dyckerhoff and M. Kapranov, *Higher Segal spaces*, Lect. Notes Math., vol. 2244, Springer, 2019.

[3] R. Exel, *Partial actions of groups and actions of inverse semigroups*, Proc. Amer. Math. Soc. 126 (1998), no. 12, 3481–3494.

[4] I. Gálvez-Carrillo, J. Kock, and A. Tonks, *Decomposition spaces, incidence algebras and Möbius inversion I: Basic theory*, Adv. Math. 331 (2018), 952–1015.

[5] P. Hackney and J. Lynd, *Partial groups as symmetric simplicial sets*, J. Pure Appl. Algebra **229** (2025), no. 2, Paper No. 107864.