Natural equivalences and weak invertibility of higher-categorical contexts

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Abstract.

A *context* in a higher category is a "pasting diagram with a hole"—an arrangement of cells around a boundary, such that inserting any compatible cell into the boundary produces a valid pasting diagram. A result expected to hold across models of higher categories is that, if a context E is built out of internal *equivalences* in a suitable way, then a well-formed equation of the form $\mathsf{E}x = b$ should have a unique solution up to equivalence. A proof of this fact exists for strict ω-categories [1], but parts of this proof do not even typecheck in a model with non-strict units.

We study this problem [2] in the setting of diagrammatic sets, which capture the "raw" combinatorics of pasting diagrams. The notion of equivalence in terms of *coinductive weak invertibility* can be formulated internally to any diagrammatic set. Diagrammatic sets support models of (∞, n)-categories, as well as faithful nerve functors from a variety of higher-categorical structures, so results proved for diagrammatic sets have wide model-independent applicability.

We introduce a coinductive relation $\mathsf{C} \simeq \mathsf{D}$ of *natural equivalence* between contexts in a diagrammatic set, and prove that it is a congruence with respect both to composition of contexts and to pasting in codimension 1. Moreover, we prove that certain natural equivalences involving units and unitors exist in every diagrammatic set. These facts give us access to a powerful algebraic calculus of natural equivalences of contexts. Finally, we study the class of contexts built out of weakly invertible diagrams, which we call *weakly invertible contexts*, and prove our main theorem: every weakly invertible context E admits a two-sided weak inverse, that is, a weakly invertible context E^* such that $\mathsf{E}\mathsf{E}^* \simeq -$ and $\mathsf{E}^*\mathsf{E} \simeq -$, where $-$ denotes the trivial context. This implies the result that we were aiming for: every well-formed equation $\mathsf{E}x = b$ admits the weakly unique solution $x \coloneqq \mathsf{E}^* b$.

References

[1] D. Ara, A. Burroni, Y. Guiraud, P. Malbos, F. Métayer, and S. Mimram. *Polygraphs: from rewriting to higher categories.* London Mathematical Society Lecture Note Series, Cambridge University Press, 2025.

[2] C. Chanavat and A. Hadzihasanovic. Equivalences in diagrammatic sets. Online preprint arXiv:2410.00123, 2024.