Generalised ultracategories and conceptual completeness of geometric logic (work in progress)

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Abstract.

Conceptual completeness is an important result in first order categorical logic, it states that, with an additional structure on the category of models of coherent first-order theory, describing categorically the notion of ultraproduct, it is possible to establish a syntax semantics equivalence between a certain class of functors called ultrafunctors from the category of models of a coherent theory to Set, and a pretopos (which can be thought of as a completion of the syntactic category of this theory). Towards showing this result Makkai introduce the notion of ultracategories [1] and ultrafunctors, these are categories equipped with an ultraproduct functor (together with data and coherence). More recently Lurie [2] reintroduced ultracategories and showed a version of conceptual completeness stating that there is an equivalence between a certain class of functors called left ultrafunctors from the category of points of a topos (models of the coherent theory classified by this topos) to Set and the Topos itself.

We want to extend Makkai and Lurie's results to any topos with enough points, the first obstruction is that the category of points of such toposes do not have a canonical notion of ultraproduct. Toward this, we introduce a new notion of generalised ultracategories, where the ultraproduct may not exist, but we may find instead the "representable" at the ultraproduct. It turns out that generalised ultracategories are connected to topological spaces the same way usual ultracategories are connected to compact Hausdorff spaces. We can assign to every topological space a unique generalised ultrastructure coming from the notion of ultrafilter convergence. We can expand on this, and show that the generalised ultrastructure of any generalised ultractegory can be fully determined by "covering" such category with suitable categories of points of topological spaces. There is a similar result for Toposes with enough points, stating that they can be expressed as the 2-colimit of certain "logical" topological groupoids [3]. We relate these facts together to deduce a conceptual completeness theorem stating that for any two toposes with enough points E and $E^{'}$ there is an equivalence between $\mathrm{Geom}(E, E^{'})$ and $\mathrm{Left}-\mathrm{ultrafunctors}(\mathrm{Points}_E, \mathrm{Points}_{E^{'}})$. Here Left ultrafunctors are the generalised version of left ultrafunctors that reduces to the ordinary ones in the case of coherent toposes. This result reduces to a one similar to Lurie's one if we replace the topos $E^{'}$ by the classifying topos of the theory of objects.

References

[1] M. Makkai, *Strong conceptual completeness for first-order logic*, Annals of Pure and Applied Logic, 40(2):167–215, 1988.

[2] J.Lurie, *Ultrcategories*, Preprint available at https://people.math.harvard.edu/ lurie/papers/Conceptual.pdf, 2018.

[3] C. Butz and I. Moerdijk, *Representing topoi by topological groupoids,* J. Pure Appl. Algebra, vol.130, no. 3, pp. 223–235, 1998.