Notions of Cauchy (co)completeness for normed categories

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Abstract.

Lawvere's famous 1973 paper [3] has inspired numerous works connecting metric theory and category theory. One of the lesser known suggestions hinted at in [3] is the notion of *normed category*: a category X where each hom-set $\mathsf{X}(A,B)$ comes equipped with a *norm* $\mathsf{X}(A,B) \to [0,\infty]$, subject to suitable axioms. Normed categories can be viewed as categories enriched in the closed symmetric monoidal category of *normed sets*, briefly mentioned in [3] and explicitly described in [1].

In continuation of Walter Tholen's talk at CT2024 (see also [2]), we also consider norms which take values in a quantale $\mathcal{V}$ (and not only in $[0,\infty]$) and recall basic properties of the category of $\mathcal{V}$-normed sets and of $\mathcal{V}$-normed categories, respectively, as well as the notions of normed colimit and Cauchy cocompleteness for $\mathcal{V}$-normed categories. We illustrate these notions with various examples of (large) normed categories. Our main focus is on the calculus of normed distributors. Besides the description of normed colimits as weighted colimits for a certain class of distributors, we investigate the classic notion of "Cauchy completeness *à la* Lawvere", that is: we call a normed category *Lawvere complete* whenever every left-adjoint normed distributor into it is representable. In particular, we provide a characterisation of Lawvere complete normed categories via an idempotent completeness condition and an approximation condition. Finally, we relate these notions to the corresponding concepts for $\mathcal{V}$-enriched categories and for ordinary categories.

References

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[2] M. M. Clementino, D. Hofmann, and W. Tholen, *Cauchy convergence in V-normed categories*, Advances in Mathematics (accepted), (2025), arXiv:2404.09032 [math.CT].

[3] F. W. Lawvere, *Metric spaces, generalized logic, and closed categories*, Rendiconti del Seminario Matemàtico e Fisico di Milano 43 (1973), 135–166. Republished in: Reprints in Theory and Applications of Categories, No. 1 (2002), 1–37.