Internal categories, algebraic model structures and type theory

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Abstract.

Internal category theory has witnessed an increase in interest recently, partially due to its connection with double category theory, 2-dimensional universal algebra and alternative foundations of mathematics. A natural line of enquiry is to investigate to what extent results from (small) category theory carry over to internal category theory. In this vein, Everaert, Kieboom and Van der Linden have shown that the canonical model structure on **Cat** (where weak equivalences are equivalences of categories and fibrations are isofibrations) can be generalised to $\mathbf{Cat}(\mathcal{E})$, where $\mathcal{E}$ satisfies suitable assumptions [2].

In this talk I will explain some recent work [4] in which the aforementioned model structure on $\mathbf{Cat}(\mathcal{E})$ is upgraded to an algebraic model structure [6]. To prove that such an algebraic model structure exists on **Cat** is not too difficult, but to unwind the explicit characterisation of this is more subtle and is both easier to do and more general if we work in $\mathbf{Cat}(\mathcal{E})$. This involves giving a characterisation of all the classes of maps involved— for example, the algebraic fibrations are given by cloven isofibrations. This falls out from the construction of the (co)monads involved which equip algebraic structure to any map freely, and translations between different types of algebraic structure.

Given a finitely complete and cocomplete 2-category $\mathcal{K}$, there is a model structure on it by defining the weak equivalences and fibrations representably [5]. Thanks to work by Bourke [1], it is possible to leverage our result and give a list of elementary axioms on a 2-category $\mathcal{K}$ such that we can upgrade this model structure to an algebraic model structure.

As an application, I will explain how we can use this to construct an algebraic internal groupoid model of Martin-Löf type theory. This recovers Hofmann and Streicher's groupoid model [3] when working with groupoids internal to **Set** and forgetting the algebraic structure. To conclude, I will apply these results to different examples of $\mathcal{E}$ such as categories of (pre)sheaves, the effective topos and categories which themselves model Martin-Löf type theory. In some cases, this gives an improved perspective on results that have been previously considered in the literature; in other cases, it gives novel results.

References

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