A double barreled approach to composing dynamical systems and their morphisms

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Abstract.
Motivated by developing a representability theory for open dynamical systems, we've investigated a *double barreled* approach to *loose* bimodules between double categories.

Joyal's "barrels" present bimodules $M : A^{\mathsf{op}} \times B \to \mathsf{Set}$ between categories as functors $p : C \to \Delta[1]$ into the walking arrow for which $p^{-1}(0) = A$ and $p^{-1}(1) = B$ and $p^-1(0 \to 1) = \bigsqcup_{a,b} P(a, b)$; C is the *collage* of the bimodule. When categorifying to double categories, we have two choices for generalization; we could look at barrels over the loosely discrete double category $t\Delta[1]$ consisting of one *tight* arrow, which would correspond to Paré-style bimodules $\mathbb{A}^{\mathsf{op}} \times \mathbb{B} \to \mathbb{S}\mathsf{pan}(\mathsf{Set})$, or we could look at barrels over the tightly discrete double category $\ell\Delta[1]$ which would correspond to *loose* bimodules — bimodules whose "heteromorphisms" go in the loose (a.k.a. pro) direction. In this talk we'll investigate the latter.

We may easily define a 2-category of loose bimodules as the slice $\mathcal{D}\mathsf{bl} \downarrow \ell\Delta[1]$, where $\mathcal{D}\mathsf{bl}$ is the 2-category of double categories, pseudo-functors, and tight transformations. We then investigate the functoriality of *restriction* of loose bimodules by constructing a product-preserving pseudo-functor

$$\mathsf{Res} : \mathcal{N}\mathsf{iche} \to \mathcal{D}\mathsf{bl} \downarrow \ell\Delta[1]$$

where $\mathcal{N}\mathsf{iche}$ is a 2-category of "niches" defined as the pullback:

$$\begin{array}{ccc}
\mathcal{N}\mathsf{iche} & \xrightarrow{\quad\quad\quad\quad} & \mathcal{D}\mathsf{bl} \downarrow \ell\Delta[1] \\
\downarrow & & \downarrow{\scriptstyle (d_1^*, d_0^*)} \\
2\mathcal{C}\mathsf{at}^{\mathsf{colax,conj}}(\Delta[1], \mathcal{D}\mathsf{bl}) \times 2\mathcal{C}\mathsf{at}^{\mathsf{colax,comp}}(\Delta[1], \mathcal{D}\mathsf{bl}) & \xrightarrow[d_0^* \times d_0^*]{} & \mathcal{D}\mathsf{bl} \times \mathcal{D}\mathsf{bl}
\end{array}$$

where the 2-categories in the bottom left are colax commuting squares in $\mathcal{D}\mathsf{bl}$ whose colaxitors are conjoint (resp. companion) *commuter* transformations in the sense of Paré. As a corollary, we conclude (using Arkor-Bourke-Ko's wonderful *symmetry of internalization*) that the restriction of a symmetric monoidal loose bimodule M by lax symmetric monoidal double functors F_0 and F_1 yields a symmetric monoidal loose bimodule $M(F_0, F_1)$ so long as the laxitors of F_0 are conjoint commuter cells and those of F_1 are companion commuter cells.

We'll end by using the above to construct a number of symmetric monoidal loose right modules of *open dynamical systems* acted upon by symmetric monoidal categories of *composition processes*, generalizing constructions of Schultz, Spivak, and Vasilakopoulou to include morphisms of systems with different interface and laying the ground for a representable account of system behavior. These constructions can be performed *pseudo-functorially* using the general span construction for Haugseng-Hebestreit-Linskens-Nuiten *adequate triples*. To account for monadic nondeterminism, we show that an suitable 2-category of adequate triples has Kleisli objects. The resulting span construction generalizes Leinster's construction of T-spans to suitable monads on adequate triples.