Positively closed topos-valued models

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Abstract.

This talk is about models of coherent theories (T) internal to Grothendieck toposes ($\mathcal{E}$).

One possible motivation is that certain classical notions of mathematics can be captured by internal models. For example, if T is the theory of local rings and $\mathcal{E} = Sh(X)$ for some space X, then T-models in $\mathcal{E}$ are locally ringed spaces with underlying space X. So if we can lift positive model theory to the topos-valued setting, then we gain new tools to study such objects.

A model of T internal to $\mathcal{E}$ is the same as a coherent functor $\mathcal{C}_T \to \mathcal{E}$, where $\mathcal{C}_T$ is the syntactic category. As a result, topos-valued positive model theory can be seen as the study of coherent functors $\mathcal{C} \to \mathcal{E}$ where $\mathcal{C}$ is some coherent category.

A key concept in positive model theory is the notion of a positively closed model. A model M is positively closed if every outgoing homomorphism reflects the validity of positive existential formulas. In the functorial language: if every outgoing natural transformation from $M : \mathcal{C} \to \mathcal{E}$ to some other coherent functor $N : \mathcal{C} \to \mathcal{E}$ is mono-cartesian. This is a global notion, as it concerns many other models.

A coherent functor $M : \mathcal{C} \to \mathcal{E}$ is strongly positively closed if for any monomorphism $u \hookrightarrow x$ in $\mathcal{C}$ we have $Mx = Mu \cup \bigcup_{v \hookrightarrow x:\ v \cap u = \emptyset} Mv$. This is a local notion as it depends only on M. It is well-known that for $\mathcal{E} = \mathbf{Set}$ the positively closed and the strongly positively closed models coincide.

Lurie's [3, Lecture 16X, Theorem 11] says that if $\mathcal{C}$ is coherent with finite disjoint coproducts, then any $\mathcal{C} \to \mathbf{Set}$ regular functor can be factored uniquely as $\mathcal{C} \xrightarrow{M} Sh(B, \tau_{coh}) \xrightarrow{\Gamma} \mathbf{Set}$ where B is a Boolean algebra, τ_{coh} is the topology formed by finite unions, M is coherent and Γ is global sections (see also [2, Theorem 4.6]). In particular a $Sh(B, \tau_{coh})$-valued model can be uniquely recovered from its global sections. This makes $\mathcal{E} = Sh(B, \tau_{coh})$ a good test case.

I will give examples of positively closed but not strongly positively closed $Sh(B, \tau_{coh})$-valued models where B is a complete Boolean algebra.

I will prove that if B is complete, $Sh(B, \tau_{coh})$-valued models satisfy some form of compactness: they can realize types. This is a major ingredient for proving that although for such models positively closed is not equivalent to strongly positively closed, it is equivalent to an alternative local property.

The talk is based on [1].

References

[1] K. Kanalas, Positively closed parametrized models, preprint arXiv:2409.11231, 2024.

[2] K. Kanalas, *$Sh(B)$-valued models of (κ, κ)-coherent categories*, Appl. Categ. Structures 33 (2025), no. 12.

[3] J. Lurie, lecture notes for categorical logic, URL = https://www.math.ias.edu/ lurie/278x.html, 2018.