Compact, Hausdorff and locally compact locales in toposes.

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Abstract.

Frames that are internal in a presheaf topos are presheaves of frames satisfying further conditions [2], C 1.6.9. For example, the internal completeness of the frame L is captured by the existence, for each $f\colon b \to a \in \mathcal{C}$, of a left adjoint $\Sigma_f \dashv Lf\colon La \to Lb$ satisfying the Frobenious law. As a result, there is a forgetful functor $U\colon \mathrm{Frm}[\mathcal{C}^{op}, \mathbf{Set}] \longrightarrow [\mathcal{C}^{op}, \mathbf{Frm}]$. It can be derived from [3] that this functor has a left adjoint $\mathfrak{z} \dashv U$ with sections $\mathfrak{z}L(a) = \{(u_f) \in \prod_{\vartheta_1 f = a} L(\vartheta_0 f) \mid \forall g\colon b \to a\ (L(g)(u_f) \leq u_{fg})\}$. This same description arises in [1] as a way to universally turn a lax Pos-enriched natural transformation into a strict one and allows us to determine the sections of internal posets of subobjects and of ideals in presheaf toposes, while it gives one leg of the equivalence between the categories of presheaves of compact Hausdorff locales and internal such locales in a presheaf topos. As this equivalence is not obtained object-wise, we investigate related conditions in terms of properties of the sections of the internal frame.

Compactness of the frame L amounts to the equalizer of the supremum, and the constantly equal to the top element, maps $idlL \to L$, being $\{L\}$. From that it can easily be derived that the sections of a compact internal frame are compact. Local compactness amounts to the existence of a map $\Lambda\colon L \to idlL$, left adjoint to the supremum map, and from that we can derive a section-wise left adjoint to the supremum map, $\lambda_a\colon La \to idl(La)$ yielding the local compactness of the sections of L. It can further be calculated that the transition maps preserve the way-below relation of the sections, equivalently the localic transition maps are finitary. Concerning the Hausdorff condition, i.e the closedness of the localic diagonal $L \to L \times L$, the preservation of coproducs by $\mathfrak{z}$ and properties of closed maps give that if each section La is Hausdorff then L is internally Hausdorff, while if $\mathfrak{z}L$ is Hausdorff so is each section La. We discuss further the sufficiency of the conditions on the sections for the internal frame to be compact or locally compact and relevant conditions for sheaf toposes. For example, for a proper inclusion of a sheaf subtopos, compactness in the smaller topos is equivalent to that in the larger topos, so one still obtains compactness section-wise.

References

[1] S. Henry, C. F. Townsend, *Compact Hausdorff locales in presheaf toposes,* Appl. Categ. Structures 31, (2023) No. 6, Paper No. 45, 11 p.

[2] P. T. Johnstone, *Sketches of an Elephant: a topos theory compendium: vol.1 and vol.2*, Oxford Logic Guides 43 and 44. Clarendon Press, Oxford, 2002.

[3] J. Wrigley, The Geometric Completion of a Doctrine, preprint arXiv:2304.07539, 2023.