Codiscrete cofibrations vs. iterated discrete fibrations for (∞, ℓ)-profunctors and ℓ-congruences

D. Kern

David Kern (`dkern@kth.se`)
KTH Royal Institute of Technology

Abstract.

The exactness properties of 2-categories [2] are formulated with simplicial kernels and codescent objects of cateads (or 2-congruences), internal categories whose underlying graph is a discrete two-sided fibration. For higher categories, this notion branches naturally into two different generalisations. On the one hand, the $(\ell-1)$-categorical fibration classifiers that should characterise $(\infty, \ell+1)$-topoi suggest looking at $(\ell+1)$-congruences (in an $(\infty, \ell+1)$-category $\mathfrak{K}$, for $\ell \in \mathbb{N} \cup \{\omega\}$) as internal categories whose underlying graphs are $(\infty, \ell-1)$-categorical two-sided fibrations, which requires working with a lax (or Gray-enriched) version of the kernel/codescent objects [3].

On the other hand, the use of lax limits and colimits can be avoided by defining ℓ-cateads as internal ℓ-categories whose underlying ℓ-graphs are iterated discrete two-sided fibrations; for this I will introduce cellular kernels and codescent objects of internal higher category objects, inspired by the formal methods of [1], making full use of the enrichment over $(\ell-1)$-categories to encode the higher structures in (strong) weighted limits. The object of this talk will then be to report on work in progress on the comparison between these two types of fibrations.

Since the $(\infty, \ell-1)$-categorical two-sided fibrations in $\mathfrak{K} = (\infty, \ell)$-$\mathfrak{Cat}$ precisely encode (∞, ℓ)-profunctors, the ideas of [4] on enriched profunctors indicate that they should correspond to codiscrete two-sided-cofibrations in $\mathfrak{K}$. In fact these will serve as the middle point between the two types of fibrations: writing $\mathfrak{Corr}_d(\mathfrak{K})$ for the ∞-category of d-iterated spans in $\mathfrak{K}$, we have a pair of adjunctions

$$\mathfrak{Corr}_1(\mathfrak{K}) \rightleftarrows \mathfrak{Corr}_1(\mathfrak{K}^{\mathrm{op}}) \leftrightarrows \mathfrak{Corr}_\ell(\mathfrak{K})$$

where the left adjunction is given by lax cocomma and comma, while the right one is given by a "bipartite" version of the cellular codescent objects and kernels, using higher commas. I will explain why, at least in $\mathfrak{K} = (\infty, \ell)$-$\mathfrak{Cat}$, the latter adjunction restricts to an equivalence between codiscrete cofibrations and iterated discrete fibrations.

References

[1] R. Betti, D. Schumacher, and R. Street, Factorizations in bicategories, Unpublished, 1999.

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[4] R. Street, *Fibrations in bicategories*, Cahiers de topologie et géométrie différentielle XXI (1980), no. 2, pp. 111–160.