The magnitude of a presheaf

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Abstract.

There is now a substantial literature on magnitude and magnitude homology [2]. The magnitude of an enriched category is the canonical invariant of its size. With different bases of enrichment, it recovers such classical invariants as cardinality, Euler characteristic, dimension, volume and surface area. Magnitude homology is a categorification of magnitude, in the sense that it is a homology theory of enriched categories whose Euler characteristic is magnitude (under hypotheses). It has been studied especially intensively for metric spaces and graphs.

But all of this existing work is on the magnitude and magnitude homology of enriched *categories*. Here, I will introduce the magnitude of enriched *presheaves*. I will explain the size-like properties of this invariant and how several existing measures of size naturally arise as instances of the general concept. In particular:

- The basic result on the magnitude of presheaves unifies elementary counting formulas for the cardinality of a colimit of sets.
- Entropy, relative entropy and conditional entropy for finite probability distributions all arise as special cases of the notion of the magnitude of a presheaf.
- Prime counting functions, in both number-theoretic and ring-theoretic contexts, also arise naturally as special cases.
- The concept of the magnitude of a presheaf allows one to give a categorical account of the PDE methods that have been successfully applied to calculate the magnitude of certain subsets of Euclidean space (as in [1], for instance).

Finally, I will introduce the dual concept, the comagnitude of a presheaf, which has an attractive interpretation in terms of the expected cardinality of the limit of a random presheaf.

References

[1] Juan Antonio Barceló and Anthony Carbery. On the magnitudes of compact sets in Euclidean spaces. *American Journal of Mathematics* 140 (2018), 449–494.

[2] Tom Leinster and Mark Meckes. Magnitude: a bibliography. Available at www.maths.ed.ac.uk/~tl/magbib, 2025.