Categories or Spaces?
Categorical Concepts in Noncommutative Geometry

W. Lowen

Violeta Borges Marques (Violeta.BorgesMarques@uantwerpen.be)
Universiteit Antwerpen

Lander Hermans (Lander.Hermans@uantwerpen.be)
Universiteit Antwerpen

Wendy Lowen (Wendy.Lowen@uantwerpen.be)
Universiteit Antwerpen

Arne Mertens (Arne.Mertens@uantwerpen.be)
Universiteit Antwerpen

Abstract.

Ever since the Grothendieck school revolutionized algebraic geometry in the 1960s, categories of sheaves have been central to the subject. Following the seminal work by Artin, Tate and Van den Bergh [1], noncommutative algebraic geometry has taken their prominence to the next level by conceiving noncommutative spaces as categories with certain algebraic generators that can be interpreted as coordinates [4] [3]. That the resulting structures are inherently geometrical is exemplified by their occurrence *in families*, that is, depending on certain parameters that can themselves be organized into so-called moduli spaces. This key feature more generally underpins Kontsevich's Homological Mirror Symmetry Conjecture which revolves around a certain exchange of associated categories for mirror manifolds [8].

In the present talk, we will discuss recent attempts to set up a corresponding *deformation theory* for categorical structures modeling spaces, thus addressing infinitesimal and formal parameters. The prime focus will be on the most relevant categorical tools in play.

We will start by reviewing the basic ideas of algebraic deformation theory in the prototypical example of algebra deformations in relation to affine schemes, as well as an extension to certain projective schemes which borrows ideas from topos theory [12] [16] [11].

Next, we will turn to an approach to deformations of more general schemes via prestacks [15]. More precisely, we will explain how a novel type of "box operadic" structure closely related to the notion of a virtual double category [10] [6] can be used to endow the deformation complex of a prestack with the desired L_∞-structure [7].

Since [8], mirror symmetry has not only been demonstrated for certain noncommutative deformations of classical schemes [2]; it also incorporates "spaces" of a rather different nature, like singularity categories which can be obtained as quotients of two different derived categories associated to a single singular scheme [17]. However, algebraic deformation theory of such categories is notoriously difficult if one uses the classical models of differential graded or A_∞-categories [9].

In this light, we will present the novel conjectural model of *quasi-categories in modules* for this type of categories [13], [14], which is inspired both by Joyal's quasi-categories and by Leinster's up-to-homotopy monoids [10]. We will show that this model is indeed amenable to algebraic deformation theory, and we will sketch some future prospects for the resulting theory [5].

References

[1] M. Artin, J. Tate, and M. Van den Bergh. *Some algebras associated to automorphisms of elliptic curves.* English. The Grothendieck Festschrift, Collect. Artic. in Honor of the 60th Birthday of A. Grothendieck. Vol. I, Prog. Math. 86, 33-85 (1990). 1990.

[2] D. Auroux, L. Katzarkov, and D. Orlov. "Mirror symmetry for weighted projective planes and their noncommutative deformations". In: *Ann. Math. (2)* 167.3 (2008), pp. 867–943. ISSN: 0003-486X.

[3] M. Van den Bergh. "Noncommutative quadrics". In: *Int. Math. Res. Not.* 2011.17 (2011), pp. 3983–4026. ISSN: 1073-7928.

[4] A. I. Bondal and A. E. Polishchuk. "Homological properties of associative algebras: The method of helices". In: *Russ. Acad. Sci., Izv., Math.* 42.2 (1993), pp. 219–260. ISSN: 1064-5632.

[5] V. Borges Marques, W. Lowen, and A. Mertens. "Deformations of quasi-categories in modules". In: *J. Pure Appl. Algebra* 229.2 (2025). Id/No 107866, p. 19. ISSN: 0022-4049.

[6] G. S. H. Cruttwell and Michael A. Shulman. "A unified framework for generalized multicategories". In: *Theory Appl. Categ.* 24 (2010), pp. 580–655. ISSN: 1201-561X.

[7] H. Dinh Van, L. Hermans, and W. Lowen. *Box operads and higher Gerstenhaber brackets.* Preprint, arXiv:2305.20036 [math.AT] (2023). 2023. URL: `https://arxiv.org/abs/2305.20036`.

[8] M. Kontsevich. "Homological algebra of mirror symmetry". In: *Proceedings of the international congress of mathematicians, ICM '94, August 3-11, 1994, Zürich, Switzerland. Vol. I.* Basel: Birkhäuser, 1995, pp. 120–139.

[9] A. Lehmann. "Hochschild cohomology parametrizes curved Morita deformations". In: *Proc. Am. Math. Soc.* 153.6 (2025), pp. 2341–2351. ISSN: 0002-9939.

[10] T. Leinster. *Higher operads, higher categories.* English. Vol. 298. Lond. Math. Soc. Lect. Note Ser. Cambridge: Cambridge University Press, 2004. ISBN: 0-521-53215-9.

[11] W. Lowen. "Grothendieck categories and their deformations with an application to schemes". In: *Mat. Contemp.* 41 (2012), pp. 27–48. ISSN: 0103-9059.

[12] W. Lowen and M. Van den Bergh. "Hochschild cohomology of Abelian categories and ringed spaces". In: *Adv. Math.* 198.1 (2005), pp. 172–221. ISSN: 0001-8708.

[13] W. Lowen and A. Mertens. *Enriched quasi-categories and the templicial homotopy coherent nerve.* Alg. Geom. Top. 2025. DOI: `10.2140/agt.2025.25.1029`.

[14] W. Lowen and A. Mertens. *Frobenius templicial modules and the dg-nerve.* Int. Math. Res. Not. 2025.

[15] Wendy Lowen. "Algebroid prestacks and deformations of ringed spaces". In: *Trans. Am. Math. Soc.* 360.3 (2008), pp. 1631–1660. ISSN: 0002-9947.

[16] Wendy Lowen. "Linearized topologies and deformation theory". In: *Topology Appl.* 200 (2016), pp. 176–211. ISSN: 0166-8641.

[17] D. O. Orlov. "Triangulated categories of singularities and D-branes in Landau-Ginzburg models". In: *Algebraic geometry. Methods, relations, and applications. Collected papers. Dedicated to the memory of Andrei Nikolaevich Tyurin.* Moscow: Maik Nauka/Interperiodica, 2004, pp. 227–248.