Fibrational approach to Grandis exactness for 2-categories

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Abstract.

The fundamental notion of exact sequence, from homological algebra, has been captured by numerous axiomatic frameworks in categorical algebra, among which the key framework of an abelian category. Our work contributes to the extension of the notion of exact sequence and of homological algebra to 2-dimensional category theory, with insights from the theory of 2-fibrations and that of factorization systems on a 2-category. Our investigation is motivated and guided by the study of the 2-dimensional exactness structures of the 2-category of abelian categories.

In an abelian category, the (bi)fibration of subobjects is isomorphic to the (bi)fibration of quotients. This property captures a substantial information about the exactness of a category. Indeed, as it was shown in [2], categories equipped with a proper factorization system such that the opfibration of subobjects relative to the factorization system is isomorphic to the fibration of relative quotients are precisely the Grandis exact categories. Explicitly, these are categories equipped with an ideal of null morphisms such that all kernels and cokernels relative to the ideal exist and every morphism factorizes as a cokernel followed by a kernel. Aiming at uncovering exactness structures of the 2-category of abelian categories, we develop a 2-dimensional analogue of the fibrational approach to Grandis exactness described above. More precisely, we explicitly characterize those $(1,1)$-proper factorization systems on a 2-category for which the weak 2-opfibration of relative 2-subobjects is biequivalent to the weak 2-fibration of relative 2-quotients.

As an outcome, we propose a 2-dimensional notion of Grandis exact category. Such a notion involves the concept of a 2-dimensional ideal of null morphisms and null 2-cells, which we reach via a profunctor approach. 2-dimensional ideals are a key element for the extension of the notion of exact sequence to dimension 2, as they naturally lead to the definition of 2-dimensional kernels and cokernels. Thanks to the results of [3], the 2-category of abelian categories, with suitably chosen functors called Serre functors as morphisms, is an example of our 2-dimensional notion of Grandis exact category.

References

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[3] Z. Janelidze and Ü. Reimaa, Serre functors between Puppe exact categories and 2-dimensional exactness, in preparation, 2025.