Homological lemmas in a non-pointed context

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Abstract.

Homological categories, namely pointed regular protomodular categories, have been shown to constitute the good context in which the non-abelian versions of the classical homological lemmas hold. Examples of homological categories are those of groups, non-unitary rings, associative algebras, Lie algebras, topological groups and many others. In particular, for pointed regular categories, protomodularity is equivalent to the validity of the short five lemma. The nine lemma holds in every homological category, and also in every regular protomodular quasi-pointed category (meaning that the unique morphism from the initial object to the terminal one is a monomorphism). Replacing short exact sequences with exact forks, it was shown in [1] that a denormalized version of the lemma holds in regular Mal'tsev categories. In [2] the authors used the framework of star-regular categories, which is based on the notion of an ideal of morphisms, for a common description of these two versions of the nine lemma. It is shown there that, in a star-regular category with "enough trivial objects", the upper and the lower nine lemmas are equivalent. Moreover, under mild additional assumptions, the middle version of the nine lemma is equivalent to a version of the short five lemma relative to stars. These results cover the known ones concerning the homological lemmas in the pointed and quasi-pointed contexts (where $\mathcal{N}$ is the class of morphisms that factor through 0) as well as the denormalized versions of them (where $\mathcal{N}$ is the class of all morphisms).

However, this context excludes several interesting examples in which some forms of the nine lemma are valid, like unitary rings, Boolean algebras, Heyting algebras, MV-algebras and, more generally, protomodular varieties of universal algebras having more than one constant. We will introduce a categorical framework which includes all the examples just mentioned, and in which suitable forms of the homological lemmas hold. We will consider regular protomodular categories $\mathcal{C}$ equipped with a full, posetal, monocoreflective subcategory $\mathcal{Z}$ of "zero objects" such that the reflector inverts monomorphisms. Pointed and quasi-pointed regular protomodular categories are examples of our situation. Another large class of examples is given by regular protomodular categories with initial object in which the unique morphism $0 \to 1$ is a regular epimorphism. This includes, in particular, *ideally exact categories* in the sense of [3], among which there are the dual categories of all elementary toposes and all protomodular varieties of universal algebras with more than one constant. Moreover, the topological models of protomodular theories with more than one constant are examples of our situation.

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References

[1] D. Bourn, *The denormalized* 3×3 *lemma*, J. Pure Appl. Algebra 177 (2003), 113–129.

[2] M. Gran, Z. Janelidze, D. Rodelo, 3×3 *lemma for star-exact sequences*, Homology Homotopy Appl. 14 n.2 (2012), 1–22.

[3] G. Janelidze, *Ideally exact categories*, Theory Appl. Categ. 41 (2024), 414–425.