Categorical logic meets double categories

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Abstract.

In this talk, we will discuss the relationship between *fibrations/doctrines*, which often appear in categorical logic, and *virtual double categories.* The key concept is the construction $\mathbb{B}\mathrm{il}$ of a virtual double category from a *cartesian fibration* (a Grothendieck fibration with finite products).

Double categories of relations. The double category $\mathbb{R}\mathrm{el}$ has sets, functions, and (binary) relations as objects, tight (vertical) arrows, and loose (horizontal) arrows, respectively. As relations between two sets are subsets of their product, this double category can be understood as arising from the subset fibration over **Set**. This is the archetype of our construction. Namely, one can construct from a cartesian fibration $\mathfrak{p}$ a virtual double category $\mathbb{B}\mathrm{il}(\mathfrak{p})$ whose loose arrows are *relations internal to the fibration* $\mathfrak{p}$. This is a generalization of the $\mathbb{F}\mathrm{r}$-construction in Shulman's work [3].

Why virtual? In $\mathbb{R}\mathrm{el}$, the identity relation and the composition of relations are given by $=$ and $\exists$, respectively. Since not all fibrations admit the interpretation of these logical symbols, we can only define $\mathbb{B}\mathrm{il}(\mathfrak{p})$ as a *virtual* double category, a structure that does not assume the loose identity or composition. Then, it is natural to suspect that this $\mathbb{B}\mathrm{il}(\mathfrak{p})$ is a double category if and only if the logic internal to $\mathfrak{p}$ has the logical symbols $=$ and $\exists$. The latter should be stated more precisely as the fibration $\mathfrak{p}$ being an *elementary existential fibration/doctrine.*

The main result. We will show that $\mathbb{B}\mathrm{il}(\mathfrak{p})$ is a double category with "fibrational structure" and "compatible finite products" precisely when $\mathfrak{p}$ is elementary existential. To refine this, we will present the 2-category of elementary existential fibrations as the pullback of that of cartesian fibrational double categories (a class of double categories with "double-categorical products" and "substitution") *[1] along the 2-functor* $\mathbb{B}\mathrm{il}$. We will also show that this restricted $\mathbb{B}\mathrm{il}$ (in the top row below) is locally an equivalence and explain how its essential image is characterized.

$$\begin{tikzcd}
\boxed{\text{Elementary existential fibrations}} \arrow[r, "\mathbb{B}\mathrm{il}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \boxed{\text{Cartesian fibrational double categories}} \arrow[d] \\
\boxed{\begin{array}{l}\text{Cartesian fibrations}\\ \text{(Fibrations with finite products)}\end{array}} \arrow[r, "\mathbb{B}\mathrm{il}"'] & \boxed{\text{Cartesian fibrational virtual double categories}}
\end{tikzcd}$$

References

[1] E. Aleiferi, *Cartesian double categories with an emphasis on characterizing spans*, PhD thesis, arXiv:1809.06940, 2018.

[2] H. Nasu, *Logical Aspects of Virtual Double Categories*, master's thesis, arXiv:2501.17869, 2025.

[3] M. Shulman, *Framed bicategories and monoidal fibrations*, Theory Appl. Categ. 20 (2008), 650–738.