Morphisms and comorphisms of sites: a double-categorical approach

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Abstract.

Geometric morphisms can be induced either from morphisms or comorphisms of sites, respectively in a contravariant and a covariant way; the first are characterized through a cover-preservation property (aside flatness), the second through a cover-lifting property. As both define a relevant notion of 1-cells between sites, one may ask whether there is a proper way to mix them altogether into a single categorical structure on sites, and if so, does it help understanding the reason for which we have those twin classes of functors rather than a single one ?

In a first part, we will explain how morphisms and comorphisms, though they do not compose with each other, can be arranged as the horizontal and vertical 1-cells of a double-category of sites. Relying on the formalism of extension-restriction applied to sieves seen as presheaves, we will then show how the sheaf-topos construction defines a horizontally contravariant, vertically covariant double-functor to the quintet double-category of topoi and discuss a few properties of this double-functor, as its relation to tabulators, or a conjoints-to-companions phenomenon.

Double-categories are a good environment to manipulate one the most expressive gadgets of category theory, the so called *exact squares* introduced by [3]: those are lax squares whose corresponding extension-restriction square is invertible; in the context of sites, one can speak more generally of *locally exact squares*, those that are sent to an invertible double cell by the sheafification double-functor. After giving an intrinsic characterization of those locally exact squares in terms of *relative cofinality* à la [1], will discuss a variety of examples in and recover several classical results of topos theory as local exactness conditions of some suited squares.

In a second part, we will discuss a reason behind this double-categorical presentation. It is known since [2] that a 2-(co)monad comes with a canonical double-category of strict (co)algebras, together with lax morphisms of (co)algebras as horizontal maps and colax morphisms as vertical maps. Here, we introduce a certain comonad sending a category to its *cofree site* containing all possible filters of sieves. Then one can exhibit sites as coalgebras for the underlying copointed endofunctor, and more crucially, cover-preserving functors as lax-morphisms of coalgebras, and cover-lifting functors as colax-morphisms of coalgebras. Modulo a few adjustments to incorporate flatness, those results give a more conceptual reason for the dichotomy between morphisms and comorphisms of sites.

References

[1] O. Caramello, *Denseness conditions, morphisms and equivalences of toposes*, 2020,

[2] M. Grandis, R. Paré, *Multiple categories of generalized quintets.*

[3] R. Guitart, *Relations et carrés exacts*, Ann. Sci. Math. Québec, vol 4, 1980.