Relational doctrines, quotient completions and projectives

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Abstract.

The notion of relational doctrine is introduce in [2] as a functorial description of the calculus of relations, very much like Lawvere's doctrines provide a functorial description of predicate logic. More specifically a relational doctrine is a contravariant functor R from the product of a category C with itself, mapping each pair (A, B) of objects of C to a poset $R(A, B)$ that is though of as the set of relations from A to B, ordered by inclusion and closed under composition, identities and the converse of a relation. Relational doctrines provide a natural setting where to deal with equivalence relations. In general relational doctrines need not have quotients and we describe a universal construction that complete a doctrine with them. We show that this construction subsumes many known examples. In particular, beside Maietti and Rosolini quotient completion of elementary and existential doctrines, we find also relational doctrines based over the category of vector and metric spaces. This last class of examples turns out to be of particular interest for the so called 'quantitative reasoning' in computer science, where one wants to interpret identity relations as distances. Since relational doctrines form a 2-category **RD** one can consider monads T over a relational doctrine R and also the corresponding Eilenberg-Moore object R^T in **RD**. We show that a relational doctrine with quotients is an instance of our completion precisely when it has enough projectives and, in this case, the doctrine of algebras R^T is the completion of its restriction to free algebras with projective generators (we compare this result with similar ones in [1, 4] and [3]).

References

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