

Categories of relations which compose independently

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Abstract.

Regular categories can be thought of as categories with a well behaved calculus of relations. In particular, they have pullbacks and image factorizations, which are used to compose relations.

In this work we study a number of categories which *look like* categories of relations, but which do not fit into the established theory of relations in regular categories. Examples are the category of Hilbert spaces and contractions, the category of sets and partial injective functions, and categories of probability spaces and stochastic maps, which one can think of as “probabilistic relations”.

Here is roughly what happens. It is well known that the canonical product of probability spaces is not a categorical product—indeed, in general products do not exist in the categories of probability theory, and neither do pullbacks. However, products of probability spaces *do* satisfy a universal property: they are universal among those spans which make their legs *conditionally independent*, in a way that we can make precise, and which can be traced back to Simpson’s work [1]. We can extend this idea from products to pullbacks (*independent pullbacks*), and use it to define a composition analogous to the one of relations. It turns out that this recovers exactly the (Markov) composition of stochastic maps [2], and the same is true for the other examples.

This work is devoted to developing a *categorical theory*, parallel to the one of regular categories, which explains these phenomena. We introduce the new notion of *epi-regular independence category*, and develop the theory of relations in these categories. In particular, we establish a strict 2-equivalence between epi-regular independence categories and *dagger categories with dilators* [3] that parallels the well-known correspondence between regular categories and tabular allegories. The dagger categories with dilators are sent to their wide subcategories of dagger-epimorphisms, and the epi-regular independence categories are sent to their associated dagger categories of relations, with composition defined through independent pullbacks and a convenient factorization.

References

- [1] A. Simpson, *Category-theoretic Structure for Independence and Conditional Independence*, Electronic Notes in Theoretical Computer Science, vol. 336, 2018.
- [2] D. Stein, *Random Variables, Conditional Independence and Categories of Abstract Sample Spaces*. Submitted, arXiv:2503.02477.
- [3] M. Di Meglio, *R^* -categories: The Hilbert-space analogue of abelian categories*. Submitted, arXiv:2312.02883.