A study of Kock's fat Delta

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Abstract.

The study of higher categories involves many tools based on the simplex category Δ and simplicial methods [5]. Indeed, Δ enables the encoding of coherences in geometric shapes for higher structures. However, the degeneracy maps in Δ encode the identity structure *strictly* in contrast to the associativity structure. The category referred to as fat Delta, denoted by $\underline{\Delta}$ and first introduced by Kock [4] as the category of relative finite semiordinals (i.e. relative finite ordinals with a total strict order relation), was developed as a means of providing a geometric interpretation of *weak* identity arrows in higher categories.

We present in [3] a comprehensive study of $\underline{\Delta}$ mainly via the theory of monads with arities [7, 2], which offers an abstract setting to produce nerve theorems and study Segal conditions. Our first main result is the nerve theorem for relative semicategories, denoted by $\mathsf{RelSemiCat}$.

Theorem 1 ([3, Theorem 4.23]). *Let* $\mathsf{RelGraph}$ *denote the category of relative directed graphs and let* $\underline{j} : \underline{\Delta}_0 \hookrightarrow \underline{\Delta}$ *be the inclusion of the wide subcategory of relative semiordinals and relative graph morphisms. The nerve functor* $\underline{\mathcal{N}} : \mathsf{RelSemiCat} \to \widehat{\underline{\Delta}}$ *is fully faithful. The essential image is spanned by the presheaves whose restriction along* $\underline{j}$ *belong to the essential image of* $\underline{\mathcal{N}}_0 : \mathsf{RelGraph} \to \widehat{\underline{\Delta}_0}$.

In particular, this indicates that $\underline{\Delta}$ is for relative semicategories what Δ is for categories. Among other consequences of the theory of monad with arities, we also show that $\underline{\Delta}$ has a special orthogonal factorisation system.

Proposition 1 ([3, Proposition 4.25]). $\underline{\Delta}$ *admits an active-inert factorisation system* $(\underline{\Delta}_a, \underline{\Delta}_0)$.

This active-inert factorisation system allows us to more easily express the Segal condition of [6, Section 4.3]. Additionally, using $(\underline{\Delta}_a, \underline{\Delta}_0)$, we can relate $\underline{\Delta}$ to Berger's theory [1]:

Theorem 2 ([3, Theorem 5.4]). $\underline{\Delta}$ *is a strongly unital and extensional hypermoment category.*

References

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