Grothendieck coverages on free monoids

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Abstract.

In [1], it is shown that the continuous actions of a topological monoid (M, τ) on sets forms a coreflective subcategory of the topos[1] of *all* actions of M (discarding the topology), and as such is a topos. The latter is equivalent to the category $\mathrm{PSh}(M)$ of presheaves on M, where M is viewed as a one-object category. Given that every topos is of the form $\mathrm{Sh}(\mathcal{C}, J)$ for some category $\mathcal{C}$ and coverage[2] J on $\mathcal{C}$, a regular question in response to [1] is:

Q: What are the coverages on M and the corresponding subtoposes of $\mathrm{PSh}(M)$ like?

All but the degenerate case are *hyperconnected* over Set, which means that the resulting toposes are 'orthogonal' to the much studied motivating case of *localic* toposes over Set, built from topological spaces (or their point-free counterparts, locales). As such, these should provide a complementary class of examples.

We consider in this talk the case where $M = \Sigma^*$ is free over a set Σ. The 'sieves' constituting a coverage J on M are *right ideals*, which can be identified with upwards-closed subgraphs of the *Cayley graph* of Σ^*, a directed tree where each vertex has children indexed by Σ. These can in turn be identified with 'prefix-independent sets' and then 'full $|\Sigma|$-ary subtrees'. We use these alternative presentations to facilitate the analysis of the possible coverages. An informal statement of this classification is as follows.

Proposition. Non-trivial, non-degenerate coverages on Σ^* are indexed by sets of equivalence classes of infinite words (elements of Σ^ω). In particular, the empty set corresponds to a minimal non-trivial coverage J_{min} such that $\mathrm{Sh}(\Sigma^*, J_{min})$ is the (generalized) Jónsson-Tarski topos, and there is a maximal non-degenerate coverage J_{max} coinciding with the dense coverage.

The equivalence relation in question is that of sharing a common (infinite) suffix. This statement is informal insofar as a countable set of equivalence classes provably determines a unique coverage, but strictly more information is needed to account for the possibilities when the set is uncountable.

A more abstract approach is to consider the Lawvere-Tierney topologies on $\mathrm{PSh}(\Sigma^*)$. Conveniently, this topos is an étendue: it has a cover by a localic topos. Even more conveniently, this topos is that of sheaves on (generalized) Cantor space. As such, we arrive at a complementary understanding of these coverages in terms of sublocales of Cantor space.

We shall end the talk by explaining how these considerations can be extended to a characterization of coverages on more general monoids.

References

[1] M. Rogers, *Toposes of Topological Monoid Actions*, Compositionality 5 (1) (2022)

[1] All toposes mentioned are Grothendieck toposes.

[2] We use the term coverage for what is usually called a Grothendieck topology. Given that we also consider topologies in the usual sense, this clash of terminology is best avoided.