Extending strong conceptual completeness through virtual ultracategories

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Abstract.

Following Makkai's *strong conceptual completeness* [1], Lurie proved in [2] a powerful reconstruction theorem: one can recover a coherent topos (seen as a theory) as sheaves over its *ultracategory* of points (seen as models). We extend this result: every topos induces a **virtual ultracategory** structure on its points, sheaves over which yield back the original topos, assuming it has enough points.

More precisely, we show the following.

Theorem. *Topoi with enough points embed reflectively into accessible virtual ultracategories.*

Intuitively, ultracategories are categories with an ultraproduct operation, and virtual ultracategories generalize ultracategories in the same way that multicategories generalize monoidal categories. The notion of virtual ultracategory fits indeed into the framework of generalized multicategories from [3], moreover ultracategories can be recovered by adding a representability condition.

Alternatively, in the same way that ultracategories are a categorification of compact Hausdorff spaces (seen as β-algebras) one can see virtual ultracategories as a categorification of topological spaces (seen as relational β-modules). The notion of virtual ultracategory takes thus naturally its place in the missing blank below.

compact Hausdorff spaces	ultracategories	coherent topoi
topological spaces	**?**	topoi with enough points

In particular, when restricted to coherent topoi, the theorem above gives a new proof of Lurie's reconstruction theorem. The proof we present relies on representations of topoi by topological groupoids, another, more established, way to reconstruct a topos from its points.

This talk is based on [4].

References

[1] M. Makkai, *Stone duality for first order logic*, Adv. Math. 65 (1987), no. 2, 97–170.

[2] J. Lurie, *Ultracategories*, www.math.ias.edu/~lurie/papers/Conceptual.pdf.

[3] G. Cruttwell, M. Shulman, *A unified framework for generalized multicategories*, Theory Appl. Categ. 24 (2010) , 580–655

[4] G. Saadia, *A reconstruction theorem for topoi with enough points*, preprint to appear.