Enhancements of quivers with relations

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Abstract.

The theory of quiver representations has long played a fundamental role in both algebra and geometry. More recently, even modern fields such as topological data analysis have benefited from insights drawn from quiver representations [1].

Let k be a field. A quiver Q is a directed graph, and a representation of Q is a functor from the quiver to the category of k-vector spaces. To study the properties of the category of representations, we consider its derived category, the category of chain complexes localized at quasi-isomorphisms—i.e., morphisms inducing isomorphisms in cohomology. However, the derived category lacks a well-behaved notion of limits and colimits, as they are not functorial. To address this, we turn to enhancements of the derived category, which provide a framework for computing homotopy limits and colimits.

One such enhancement is provided by *derivators*, introduced independently by Grothendieck, Heller, and Franke, and further developed by Groth (see [4], [2]). Heuristically, a derivator can be viewed as a collection of "homotopy categories of diagrams," making it a natural tool for studying quivers. Indeed, Groth and Šťov'ıček developed a derivator-theoretic approach to the representation theory of Dynkin quivers of type A [3].

To investigate the category of representations, we can equivalently examine the module category of the path algebra of a quiver, an algebra whose underlying module is freely generated by all paths in Q, with multiplication given by concatenation. A less-explored direction in this framework is the derivator enhancement of quivers with relations, where relations are algebraic constraints imposed on the path algebra via a quotient by an ideal. A first instance of such an enhancement was studied in [5], but a general approach to extending this construction to a wider class of relations remains open.

In this talk, we address this problem by computing homotopy Kan extensions and analyzing their behavior in this context. Our results provide a derivator enhancement for all quadratic monomial relations over Dynkin quivers of type A. A direct application of this result leads to a derivator-theoretic version of derived equivalences arising from Koszul duality, showing that these equivalences are universal, meaning they do not depend on the choice of the field of coefficients.

References

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