Mnemetic Lax Idempotent Monads and Compactness

Quentin Schroeder

Quentin Schroeder (`quentin.schroeder@etu.u-paris.fr`)
LIPN, Paris 13 University

Jonas Frey (`frey@lipn.univ-paris13.fr`)
LIPN, Paris 13 University

Abstract. Lax idempotent 2-monads (also known as *Kock-Zöberlein monads* [1, 2]) give a framework for *free cocompletions*. Examples are the *down-set monad* on posets (adjoining all joins to a poset) or the *Ind-completion* on categories (adjoining all filtered colimits). Moreover, for these monads being an algebra becomes a property making l.i. monads *property-like* [3]. The technical condition for and object $A \in \mathcal{A}$ to be an algebra of an l.i. monad $T : \mathcal{A} \to \mathcal{A}$ is that the unit $\eta_A : A \to TA$ admits a left adjoint with invertible counit.

In many of these situations, we have a notion of *compactness* or *primality*, which often allows for generators to be recovered from a cocompletion. For example, a category with finite limits can be reconstructed from its *ex-lex completion* as the full subcategory of *regular projective objects*, and any poset can be identified with the *completely join prime* elements of its suplattice completion. On the other hand, a locally small category can be recovered from its small cocompletion only up to closure under retracts (Cauchy completion), and for genuinely idempotent T there is no hope to recover A from TA.

We propose an abstract criterion to characterize l.i. monads in which A can be recovered from TA: a *mnemetic monad* is a l.i. monad for which the diagram

$$\begin{tikzcd} A \arrow[r, "\eta_A"] & TA \arrow[r, shift left=2, "T\eta_A"] \arrow[r, shift right=2, "\eta_{TA}"'] \arrow[r, phantom, "\Downarrow\theta_A"] & TTA \end{tikzcd}$$

is an *inverter* for all $A \in \mathcal{A}$, where $\theta_A : T\eta_A \to \eta_{TA}$ is the mediating 2-cell of the adjoint cylinder $T\eta_A \dashv \mu_A \dashv \eta_{TA}$.

In many cases, the adjunction arising from a l.i. monad can be decomposed into an idempotent and a mnemetic part, for example, the monadic functor $\mathsf{CoComp} \to \mathsf{CAT}$ from cocomplete to locally small categories factors through Cauchy complete categories:

$$\begin{tikzcd} \mathsf{CAT} \arrow[r, bend right] \arrow[r, phantom, "\top"] & \mathsf{CAT}_{\mathsf{cc}} \arrow[l, bend right] \arrow[r, bend right] \arrow[r, phantom, "\top"] & \mathsf{CoComp} \arrow[l, bend right] \end{tikzcd}$$

However, the naive approach to exhibit such a factorization does not work in general, the problem being that TA need not be the cocompletion of the inverter of θ_A. The talk will present a counterexample to this effect, as well as discuss notions such as *compactness* and *continuity* in the abstract framework.

References

[1] Kock, A. Monads for which structures are adjoint to units. *J. Pure Appl. Algebra.* **104**, 41-59 (1995,10)

[2] Zöberlein, V. Doctrines on 2-Categories. *Math. Z.* **148** pp. 267-280 (1976)

[3] Kelly, G. & Lack, S. On property-like structures.. *Theory And Applications Of Categories.* **3** pp. 213-250 (1997),