Principal bundles in join restriction categories

F. Schwarz

Robin Cockett (robin@ucalgary.ca)
University of Calgary

Florian Schwarz (florian.schwarz@ucalgary.ca)
University of Calgary

Abstract. Principal bundles have three different definitions, depending on the category of geometric objects you study.

In Differential Geometry, [6] defines them as a locally trivial projection map of smooth manifolds with an atlas whose transition maps are given by group multiplication. In Topology [7] defines them as G-equivariantly trivial G-spaces. In Algebraic Geometry, [8] defines them as étale locally isotrivial geometric quotients of G-varieties.

The goal of this work is to have an overarching categorical notion that recovers all of them. While they are different objects, they all have in common that they are locally isomorphic to the Cartesian product of a base space with a group.

In order to give a purely categorical definition of a principal bundle, we formulate this condition in the language of join restriction categories. Restriction categories were developed in [1, 2, 3] to generalize partial maps (maps defined only on a subset of the domain) and have since then found applications in mathematics and computer science. Join restriction categories, as described in [5] are restriction categories where local restrictions can be used to obtain a global map. Together with a manifold construction inspired by [4] that allows us to glue together objects, this setup allows the description of principal bundles entirely in the language of join-restriction categories.

Our categorical principal bundles recover the principal bundles from topology and differential geometry and we hope to also incorporate the principal bundles from algebraic geometry as an example of this in a carefully chosen partial map category of schemes.

References

[1] J. R. B. Cockett and S. Lack, *Restriction categories. I. Categories of partial maps*, Theoret. Comput. Sci. 270 (2002), no. 1-2, 223-259.

[2] J. R. B. Cockett and S. Lack, *Restriction categories. II. Partial map classification*, Theoret. Comput. Sci. 294 (2003), no. 1-2, 61-102.

[3] J. R. B. Cockett and S. Lack, *Restriction categories. III. Colimits, partial limits and extensivity*, Math. Structures Comput. Sci. 17 (2007), no 4, 775-817.

[4] M. Grandis, *Cohesive categories and manifolds*, Ann. Mat. Pura Appl. (4) 157 (1990), 199-244.

[5] X. Guo, *Products, Joins, Meets and Ranges in Restriction Categories*, PhD thesis, University of Calgary, 2012.

[6] P. W. Michor, *Topics in Differential Geometry*, Graduate studies in mathematics, American Mathematical Society, 2008.

[7] S. A. Mitchell, *Notes on principal bundles and classifying spaces*, https://sites.math.washington.edu/~mitchell/Notes/prin.pdf, 2011.

[8] G. Vooys, *Categories of Pseudocones and Equivariant Descent* preprint arXiv:2401.10172, 2024.