A Constructive Small Object Argument

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Abstract.
The small object argument is an important tool in homotopy theory: it is commonly used to construct weak factorisation systems from collections of morphisms. It was originally proved by Quillen [3], but has evolved significantly over time. In recent work, Bourke and Garner [2] proved the most general version of the small object argument thus far, using the notion of cofibrant generation by a double category to generate algebraic weak factorisation systems.

Given a double functor $U : \mathbb{J} \to \mathbb{Sq}(\mathcal{C})$ with $\mathbb{J}$ small, the proof reduces to showing that the induced functor $(\mathbb{J}^{\pitchfork})_1 \to \mathcal{C}^{\mathbf{2}}$ has a left adjoint. Whereas Bourke and Garner give an indirect proof, in this talk we will present a new proof of the small object argument in which we explicitly construct this adjoint. This results in a simpler and more transparent proof. One crucial ingredient that we introduce is the notion of a *one-step lifting structure*. We identify the initial object in the category of one-step lifting structures, and we utilize the induced universal property throughout our proof.

As an application, we show that a special case of our theorem yields a constructive version of the small object argument. This (constructively) shows that the effective Kan fibrations are the right class of an algebraic weak factorization system, thus resolving an open problem in the theory of effective Kan fibrations [1], which aims to develop a constructive model of homotopy type theory based on simplicial sets.

This talk is based on joint work in progress with Benno van den Berg and John Bourke.

References

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[3] D.G. Quillen, *Homotopical Algebra. Lecture Notes in Mathematics*, no. 43. Springer-Verlag, Berlin (1967).