Partializations of Markov categories

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Abstract.

Many operations in probability theory involve constructions such as limits or integrals that are not always defined. Even a relatively innocuous construction like the average $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i$ of a sequence (X_i) of random variables is only defined when the limit exists.

A typical manner of thinking of a "partially defined morphism" $X \to Y$ is as a "totally defined" operation defined on some subobject $D \subseteq X$. In earlier works on restriction categories such as [1], this idea is formalized, leading to a construction of a category of "partial morphisms".

We develop a construction of a CD category of "partial stochastic maps" from a particular type of Markov category called a **partializable Markov category**. These generalize the span construction of [1] to a non-deterministic/non-Cartesian setting.

Explicitly, we call a Markov category $\mathcal{C}$ **partializable** when all isomorphisms are deterministic, pullbacks of deterministic monomorphisms exist and are themselves deterministic, and deterministic monomorphisms are closed under tensoring. We show that there is then a CD category $\mathsf{Partial}(\mathcal{C})$ whose objects are those of $\mathcal{C}$ and whose morphisms are isomorphism classes of spans $X \leftarrowtail D \to Y$ with $D \rightarrowtail X$ a deterministic monomorphism. Tensoring is done leg-wise.

Our main example is the category $\mathsf{BorelStoch}$ of standard Borel spaces and stochastic maps. The morphisms $X \to Y$ in $\mathsf{Partial}(\mathsf{BorelStoch})$ can be identified with stochastic maps $D \to Y$ for a measurable $D \subseteq X$, capturing the intuition of "partially defined stochastic maps".

We characterize structures in $\mathsf{Partial}(\mathcal{C})$ like the restriction partial order, determinism and split idempotents. We also show that properties such as positivity, representability (distribution objects), conditionals, and Kolmogorov products extend from $\mathcal{C}$ to its partialization.

Given distribution objects, the distribution functor P is shown to define a monad on the subcategory of deterministic morphisms, with associated **partial algebras**. We also show that the "averaging map" assigning to a distribution p on $\mathbb{R}_{\geq 0}$ its expectation $\int x\, p(dx)$ (when finite) is such a partial algebra (on standard Borel spaces).

This is companion work to [3] on categorifying the law of large numbers. There one needs "empirical sampling morphisms", intuitively taking a sequence of points and returning a sample from its empirical distribution, which need not always be defined, hence partial at best.

Works like [2] have developed similar CD categories generalizing sub-probability measures. While similar in spirit, crucial to applications such as [3] is the determinism of *all* domain injections, excluding general sub-probability measures.

References

[1] J. R. B. Cockett and C. Lack, *Restriction categories I: Categories of partial maps*, Theoret. Comput. Sci. 270 (2002), no. 1-2, 223–259.

[2] E. D. Lavore and M. Román, *Evidential Decision Theory via Partial Markov Categories*, Ann. ACM/IEEE Symposium on Logic in Computer Science (LICS) 38 (2023), 1–14.

[3] T. Fritz, T. Gonda, A. Lorenzin, P. Perrone and A. Shah Mohammed, *Empirical Measures and Strong Laws of Large Numbers in Categorical Probability*, preprint arXiv:2503.21576, 2025.