

The Dialectica construction for comprehension categories

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Abstract.

Function variables and induction principle for Π -types. In dependent type theory extensional Σ -types and Π -types admit a *negative formulation*, as explained in the HoTT book: the focus is on the elimination rule(s), and terms of such types can be introduced if we know how to eliminate them. The Σ -type constructor also admits a *positive presentation*, where elimination is phrased via an *induction principle* (split term): the focus is on its introduction rule, and terms of a Σ -type can be eliminated as long as *canonical* terms can be eliminated. To provide Π -types with an analogous positive presentation based on an induction principle (funsplit term), we allow *function variables* to appear in a variable context, adding to the theory a suitable context formation rule [1].

Models of function variable contexts. In this talk we consider dependent type theories with function variables, formulated as *fvcccs*. An fvccc (namely, a *comprehension category of function variable contexts*) is a split cc (*comprehension category*) whose context category has, for every display map $P_A : \Gamma.A \rightarrow \Gamma$, an I_A -relative right adjoint to the weakening functor along P_A , where I_A is the inclusion of the full subcategory of the display maps over $\Gamma.A$ into the slice category over $\Gamma.A$. This weakening of the property of *local cartesian closure* is equivalent to allowing function variables to appear in the contexts of the theory associated to such a cc.

Dialectica completion for pure dependent type theories. The ordinary free constructions [4] to add Σ -types and Π -types to a Grothendieck fibration (or a logic-enriched type theory) do not preserve the structure of a comprehension category (or a pure type theory). Therefore, we describe the two pseudomonads that add *specifically to such fvcccs* the Σ -type constructor and the Π -type constructor, respectively, exploiting their positive presentation, i.e. their *universality*. In detail, the former captures the *new* dependent types as the *old* dependent contexts of the input fvccc [2]. The latter includes the function variable contexts of the input fvccc within the dependent types of the output one. We study the distributivity law between these two pseudomonads over the 2-category of fvcccs as well as the associated *Dialectica pseudomonad*. In line with our previous work [3, 4], we provide a description of the algebras of the latter and of the theories that are obtained as Dialectica completions of other theories.

References

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