# Where do ultracategories come from?

U. Tarantino

**Umberto Tarantino** (tarantino@irif.fr)
IRIF, Université Paris Cité

**Joshua Wrigley** (wrigley@irif.fr)
IRIF, Université Paris Cité

**Abstract**.

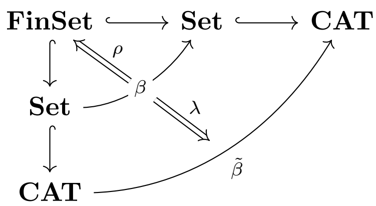
Compact Hausdorff spaces are topological spaces enjoying two surprising properties: they are algebras for a monad on **Set**, namely the ultrafilter monad $\beta\colon \mathbf{Set} \to \mathbf{Set}$, whose underlying functor is the (right) Kan extension of $\mathbf{FinSet} \hookrightarrow \mathbf{Set}$ along itself. In the same spirit, ultracategories [1, 3] are categories endowed with a "topological" structure – insofar as they allow for a Stone-like duality for first-order logic – which are proved in [5] to be algebras for a pseudomonad on **CAT**. Hamad's construction resembles Rosolini's *ultracompletion* pseudomonad from his talk at CT2024, and it heavily exploits the properties of ultrafilters in its definition. However, it is still unclear how this pseudomonad relates to $\beta$: in particular, whether it is universally-induced by $\beta$.

In this talk, we will make ultracategories emerge as algebras for a pseudomonad on **CAT** induced by $\beta$ by means of a (now *left*) Kan extension. The crucial observation is that ultracategories are algebras for a *relative* 2-monad on **CAT** (in the sense of [4]) over the inclusion $\mathbf{Set} \hookrightarrow \mathbf{CAT}$ of sets as discrete categories, whose underlying (2-)functor is $\beta : \mathbf{Set} \to \mathbf{CAT}$. Inspired by a similar result in [2] for relative (1-)monads, we will therefore introduce suitable conditions on a relative 2-monad $T$ over a 2-functor $J : \mathbf{B} \to \mathbf{CAT}$ such that:

1. the left Kan extension $\tilde{T} : \mathbf{CAT} \to \mathbf{CAT}$ of $T$ along $J$ carries the structure of a pseudomonad;
2. $T$-algebras are equivalent to $\tilde{T}$-algebras.

In the case of $\beta$, this yields the following.

**Theorem.** Ultracategories are algebras for a pseudomonad on **CAT** whose underlying 2-functor $\tilde{\beta}\colon \mathbf{CAT} \to \mathbf{CAT}$ is the unique 2-functor such that $(\tilde{\beta}, \lambda)$ is a left Kan extension and $(\beta, \rho)$ is a right Kan extension in the diagram on the right.




## References

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[5] A. Hamad, Ultracategories as colax algebras for a pseudo-monad on CAT, preprint arXiv:2502.20597, 2025.