Enriched categorical logic and accessibility

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Abstract.

In this talk I will discuss parts of the project that Jiří Rosický and myself are undertaking through the mostly unexplored field of *enriched categorical logic*. On the one hand, this is motivated by the desire of capturing under the same theory several notions of logic already developed for specific instances of enrichment: over the categories **Pos** of posets, ω-**CPO** of posets with ω-directed joins, and **Met** of metric spaces (all examples that are relevant in computer science). On the other hand, we wanted to explore applications for enrichment over the categories **Cat** of small categories, **sSet** of simplicial sets, **Ab** of Abelian groups, **DGAb** of differentially graded Abelian groups, and **Ban** of Banach spaces.

Among our goals is to define constructive and effective ways to build examples of *enriched accessible categories*; a certain kind of well-behaved enriched categories that satisfy very useful properties, but that can sometimes be difficult to recognise. Insofar, for a suitable base of enrichment $\mathcal{V}$, we have been able to exhibit specific fragments of enriched logic whose $\mathcal{V}$-categories of models describe the following subclasses of accessible $\mathcal{V}$-categories: finitary (as well as infinitary) varieties over $\mathcal{V}$ $(s=t)$[1], locally presentable $\mathcal{V}$-categories $(s=t, R(t), \wedge, \exists!)$[2], enriched injectivity classes $(s=t, R(t), \wedge, \exists)$[3], and enriched cone-injectivity classes $(s=t, R(t), \wedge, \exists, \vee)$[2].

While such presentations are in spirit very similar to those occurring in ordinary category theory, to prove the results above one needs to overcome several obstacles that are specific to the enriched setting. These, rather than the actual characterization theorems, will be the objectives of my talk. In particular, without assuming any significant background on enriched categories or logic, I will focus on the following problems: (1) what it means for an $\mathbb{L}$-structure to present a formula, (2) what rules of ordinary logic fail in this context, and (3) how existential quantification and disjunctions are interpreted over different bases (and produce interesting results).

[1] https://arxiv.org/abs/2310.11972
[2] to appear
[3] https://arxiv.org/abs/2406.12617