Arrow algebras

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Abstract. A *locale* L is a complete poset in which the following distributive law holds:

$$a \wedge \bigvee_{b \in B} b = \bigvee_{b \in B} a \wedge b,$$

when $a \in L$ and $B \subseteq L$. Whenever you have a locale, you can obtain a topos from it by taking the category of *sheaves* over the locale: the result is called a *localic topos*. This category of sheaves over the locale L is equivalent to a category that has a description in terms of logic. Indeed, there is an equivalent category of L-sets, which are sets with an L-valued equality relation on them, where this equality relation is required to be symmetric and transitive; the morphisms of L-sets are L-valued functional relations.

The latter category can be understood as the result of a two-step process. First, one builds a *tripos* out of the locale L and then one turns this tripos into a topos by the *tripos-to-topos construction*. Importantly, there are triposes that do not arise from locales, for instance, the effective tripos, whose associated elementary topos is Hyland's effective topos, a non-localic (even non-Grothendieck) topos.

The aim of this talk is to introduce *arrow algebras* and explain the work of my former MSc students Marcus Briët and Umberto Tarantino [1, 3]. Arrow algebras are algebraic structures generalising locales. The point is that they still allow you to construct a tripos, an *arrow tripos*, and hence also an *arrow topos* by the tripos-to-topos construction. In this way arrow algebras are similar to Alexandre Miquel's *implicative algebras* [2], which they generalise.

These arrow toposes include the localic toposes, but also Hyland's effective topos. Indeed, many realizability toposes can be shown to be arrow toposes, because every *pca* (*partial combinatory algebra*) gives rise to an arrow algebra: this includes also "relative, ordered" pcas as in, for example, Jetze Zoethout's PhD thesis.

Crucially, Umberto Tarantino has developed a notion of morphism of arrow algebras which correspond to geometric morphisms between the associated triposes. This has allowed us to understand the following in purely arrow algebraic terms:

1. Every arrow morphism factors as a surjection followed by an inclusion, inducing the corresponding factorisation on the level of triposes and toposes.

2. Every subtripos of an arrow tripos coming from an arrow algebra L is induced by a *nucleus* on L. Given this nucleus, there is a simple construction of a new arrow algebra inducing the subtripos.

As a result, arrow algebras provide a flexible framework for constructing and studing new toposes.

References

[1] M. Briët and B. van den Berg. *Arrow algebras*, 2023. arXiv:2308.14096.

[2] A. Miquel. *Implicative algebras: a new foundation for realizability and forcing*, Math. Struct. Comput. Sci., volume 30, number 5, 2020, pages 458–510.

[3] U. Tarantino. *A category of arrow algebras for modified realizability*, Theory and Applications of Categories (44), 2025, pages 132-180.