Cocompleteness of synthetic $(\infty, 1)$-categories

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Abstract. Riehl and Shulman [RS17] introduced *simplicial type theory* (STT) to reason about ∞-categories (meaning $(\infty, 1)$) synthetically. In our version of their theory, we work with a 0-type Δ^1 that is a bounded distributive totally ordered lattice, giving rise to all simplices Δ^n as well as their boundaries and horns. A type C is then a(n) (∞-)*category* if the induced maps $C^{\Delta^2} \to C^{\Lambda_1^2}$ and $C \to C^{\mathbb{E}}$ are equivalences (here, $\mathbb{E}$ is the free bi-invertible arrow). A type C is a(n) (∞-)*groupoid* or *space* if $C^{\Delta^1} \to C$ is an equivalence. Adding modalities from Lawvere's cohesion (discrete (co)reflection, discrete reflection, localization at Δ^1) and category theory (opposite and twisted arrow types), we gave an account to the category of spaces $\mathcal{S}$ and the universal left fibration $\pi : \mathcal{S}_* \to \mathcal{S}$, showing that $\mathcal{S}$ is directed univalent [GWB24]. Using the twisted arrow modality, we proved the Yoneda lemma for $\mathcal{S}$-valued presheaves, and developed first steps in presheaf theory and Kan extensions [GWB25]. A category C is *cocomplete* if $C \to C^I$ is a right adjoint for all small $I :_\flat \mathcal{U}$. Our main results offer simpler alternative conditions for a category to be cocomplete.

Theorem 1. *A category C is cocomplete if and only if any of the following hold:*

1. *C has finite coproducts and all sifted colimits.*
2. *C has all finite colimits and all filtered colimits.*

In the above, filtered and sifted colimits are defined using the notion of cofinal maps as introduced in STT by [GWB25] and closely follow the standard definitions from ∞-category theory: A category C is sifted if $C \to C^n$ is right cofinal for all $n : \mathbb{N}$ and filtered if $C \to C^K$ is right cofinal for all finite complexes K. We apply the above results to the category of *spectra* Sp. If (∞-)groupoids replace the category of sets in ∞-category theory, spectra take the place of abelian groups. We are now able to show that Sp is stable (Theorem 2, (2)) and use this to construct homology theories satisfying the Eilenberg–Steenrod axioms. Sp is given by the (HoTT) limit $\varprojlim(\mathcal{S}_* \xleftarrow{\Omega} \mathcal{S}_* \xleftarrow{\Omega} \ldots)$.

Lemma 2.
1. Sp *has all filtered colimits and all limits.*
2. Sp *is finitely (co)complete, $\mathbf{0}_{\mathsf{Sp}} \cong \mathbf{1}_{\mathsf{Sp}}$, and pushouts and pullbacks coincide.*

Corollary 3. Sp *is cocomplete.*

References

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