Metric spaces, entropic spaces and convexity

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Abstract.

A certain notion of convexity of sets can be captured by a monad, known as a convexity monad or barycentric monad; this is a finite version of so-called probability monads. Various authors (including Mardare, Panangaden and Plotkin [4], and Fritz and Perrone [2]) have looked at convexity/probability monads on categories of metric spaces. The work of Fritz and Perrone can be recast in terms of enriched categories if you consider metric spaces as $\overline{\mathbb{R}}_+$-categories, that is, as categories enriched over the quantale of extended non-negative real numbers $\overline{\mathbb{R}}_+$.

One can then do a similar thing for any 'suitably convex' quantale R and define a convexity monad on the category of R-categories. In particular, if we consider the quantale $\overline{\mathbb{R}}$, the extended real line $[-\infty, \infty]$ with the opposite order to that used in metric spaces, then $\overline{\mathbb{R}}$-categories are what Lawvere [3] called 'entropic spaces' and argued gave a necessary structure for state spaces in thermodynamics. The category of strict algebras with lax algebra maps for the convexity monad in this case is the category of convex entropic spaces with concave maps. The hope is that this connects Lawvere entropic approach to thermodynamics with the approach of Baez, Lynch and Moeller [1] which involves convex spaces and concave maps.

References

[1] John C. Baez, Owen Lynch, Joe Moeller, *Compositional Thermostatics*, `https://arxiv.org/abs/2111.10315`

[2] Tobias Fritz and Paolo Perrone, *A probability monad as the colimit of spaces of finite samples*, Theory and Applications of Categories, Vol. 34, 2019, No. 7, pp 170-220. `http://www.tac.mta.ca/tac/volumes/34/7/34-07.pdf`

[3] F. William Lawvere, *State categories, closed categories, and the existence of semi-continuous entropy functions*, IMA Preprints Series Series 86 (1984).

[4] Radu Mardare, Prakash Panangaden, and Gordon Plotkin. 2016. *Quantitative Algebraic Reasoning.* In Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science (LICS '16). Association for Computing Machinery, New York, NY, USA, 700–709. `https://doi.org/10.1145/2933575.2934518`