Locales are dense in Toposes

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Abstract.

It is by now a well-established fact that Grothendieck's generalised spaces, *toposes*, can be represented by their lower-dimensional analogues, *locales*. Indeed, in their seminal paper [2], Joyal and Tierney showed that every topos can be expressed in terms of a localic groupoid, and this line of work was then taken up by many other authors.

These results tell us that each topos $\mathcal{E}$ can be built out of a well-chosen finite amount of localic data. But one may wonder, what if one took a more global point of view and tried to recover $\mathcal{E}$ from all of its *canonical* localic data? That is to say, is the subcategory of locales *dense* in the category of toposes?

In this talk, we answer this question in the affirmative by showing that every topos $\mathcal{E}$ can be recovered from its "functor of *localic points*" $\mathrm{LPts}(\mathcal{E})$, which is a geometric *stack* over the category of locales as shown in [1]. The category $\mathrm{LPts}(\mathcal{E})_X$, for a locale X, is that of geometric morphisms $\mathrm{Sh}(X) \to \mathcal{E}$. In other words, $\mathrm{LPts}(\mathcal{E})$ consists of all the generalised points of $\mathcal{E}$ with localic domain.

Precisely, we show that the 2-category of toposes embeds fully faithfully via LPts in the 2-category of stacks over locales.

$$\mathrm{Topos} \xhookrightarrow{\mathrm{LPts}} \mathrm{Stack}(\mathrm{Loc})$$

This answers the question of density as LPts is exactly the nerve of the inclusion of locales in toposes. The proof relies on the stackiness of $\mathrm{LPts}(\mathcal{E})$, the description of stackification in [1], and the covering theorem of [2].

As a consequence of this result, one may now define a notion on locales and extend it to toposes "by continuity" in a canonical fashion. Time allowing, we will illustrate this principle.

References

[1] Marta Bunge. "An application of descent to a classification theorem for toposes". In: *Math. Proc. Camb. Philos. Soc.* 107, No. 1, 59-79 (1990). DOI:10.1017/S0305004100068365

[2] André Joyal and Myles Tierney. "An extension of the Galois theory of Grothendieck". In: *Mem. Amer. Math. Soc.* 51.309 (1984), pp. vii+71. DOI: 10.1090/memo/0309.