CT2025

Book of Abstracts

Brno, Czech Republic

13th – 19th of July, 2025

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	Monday		
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15:00	**Rogers** Grothendieck coverages on free monoids	**Jaz Myers** A double barreled approach to composing dynamical systems and their morphisms	**Li** Extensional concepts in intensional type theory, revisited
15:30	**Bruske** Compactological Spaces: Constructing condensed mathematics from classical topology	**Aravantinos-Satiropoulous** Fundamental Properties of Monads in Double Categories	**Spadetto** The Dialectica construction for comprehension categories
16:00	Coffee		
16:30	**Vasilakopoulou** The commuting tensor product of multicategories	**Ferguson** Monilmorphisms and relative extensivity	**Doña Mateo** An adjunction between categories of monads
16:40	**d'Espalungue** Category Theory within a 2-Category: Internal Enrichment, Presheaf Objects and Convolution Products	**Grossman** Pretorsion Theories on $(\infty,1)$-Categories	**Lanfranchi** The formal theory of vector fields for tangentads
16:50	**Nasu** Categorical logic meets double categories	**Campanini** Building pretorsion theories from torsion theories	**Theart** Extensive Morphisms
17:00	**Siqueira** Double functorial representation of indexed monoidal structures	**Mancini** Weak action representability and categories of algebras	**Schroeder** Mnemetic Lax Idempotent Monads and Compactness
17:10	**Kern** Codiscrete cofibrations vs. iterated discrete fibrations for (∞,ℓ)-profunctors and ℓ-congruences	**Reimaa** An intrinsic approach to kernels in general categories	**Sava** Enhancements of quivers with relations
17:20	**Femic** Premonoidal and Kleisli double categories	**Duvieusart** The monad of factorizations and its decomposition	**Belmonte** Lax Monoidal Structures from Monoidal Structures

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10:00	**Clarke** A new framework for limits in double categories		
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11:30	**Lowen** Categories or Spaces? Categorical Concepts in Noncommutative Geometry		
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15:00	**Saadia** Extending strong conceptual completeness through virtual ultracategories	**Montoli** Homological lemmas in a non-pointed context	**Brown** Weights for oplax colimits
15:30	**Yuksel** Locales are dense in Toposes	**Egner** Action representability of internal 2-groupoids	**Jelínek** Bicolimit Presentations of Type Theories
16:00	Coffee		
16:30	**Hamad** Generalised ultracategories and conceptual completeness of geometric logic (work in progress)	**Mattenet** The cohomology objects of a semi-abelian variety are small	**Offord** Eckmann-Hilton Arguments in Weak ω-categories
16:40	**Lindan** Exploring dualities beyond sound doctrines	**Forsman** In Regular Protomodular Categories with an Initial Object, Noetherian Objects are Hopfian and Closed Under Subobjects, Quotients and Extensions	**Benjamin** Naturality in Weak ω-categories
16:50	**Desrochers** Extending an arithmetic universe by an object	**Culot** Projective crossed modules in semi-abelian categories	**Hadzihasanovic** Natural equivalences and weak invertibility of higher-categorical contexts

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11:00	**Chorny** Homotopical recognition of diagram categories
11:30	**Stenzel** The higher algebra of monoidal bicategories
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	Divadlo Bolka Polívky		
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10:30	Coffee		
11:00	**Rodelo** Ord-Mal'tsev categories		
11:30	**Mesiti** Fibrational approach to Grandis exactness for 2-categories		
12:00	**Caviglia** Structures for the category of torsion theories		
12:30	Lunch		
	Room B2.13	Room B2.23	Room B2.43
14:30	**Ching** Closed categories, pro-operads and Goodwillie calculus	**Willerton** Metric spaces, entropic spaces and convexity	**Pasquali** Relational doctrines, quotient completions and projectives
15:00	**Abellán** $(\infty, 2)$-Topoi and descent	**Shah-Mohammed** Parializations of Markov categories	**Moreau** Profinite completions and clones
15:30	**Blom** Lax idempotent monads in homotopy theory	**Zorman** Infinite and Non-Rigid Reconstruction Theory	**Bevilacqua** Properties of coalgebraic models of a Lawvere theory
16:00	Coffee		
16:30	**Karazeris** Compact, Hausdorff and locally compact locales in toposes	**Kruml** Categories for industrial planning	**Kanalas** Positively closed topos-valued models
16:40	**Prezado** Unified approach to pointfree T_0-spaces	**Kalugins** Representables in fuzzy category theory	**Capucci** 2-classifiers for 2-algebras
16:50	**Hora** Topoi of Automata	**Opršal** A categorical perspective on the complexity of satisfying constraints	**Feierabend** Confluence of Term Rewriting Systems with Variable Binding
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9:00	**Sarazola** Double categorical equivalences		
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11:00	**Rosický** Stability from the categorical point of view		
11:30	**Tendas** Enriched categorical logic and accessibility		
12:00	**Fujii** Enrichment and families over virtual double categories		
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14:30	**Vooys** Immersions, Submersions, Local Diffeomorphisms, and Relative Cotangent Complexes in Tangent Categories	**Seip** A Constructive Small Object Argument	**Weinberger** Cocompleteness of synthetic $(\infty, 1)$-categories
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15:30	**Garcia-Martinez** Categorical-algebraic characterisations of Lie algebras	**Cox** A new tool for detecting cofibrant generation	**Hackney** Actions of partial groups, and the higher Segal conditions
16:00	Coffee		
16:30	**Chanavat** Gray products of diagrammatic (∞, n)-categories	**Luckhardt** Giry monad revisited	**Das** Schemes relative to Actegories
16:40	**Doherty** Cartesian monoidality of the cubical Joyal model structure	**Pasqualone** Posites: The Foundation of Factorization Homology	**Aristote** Open power-objects in categories of algebras
16:50	**Pradal** A study of Kock's fat Delta	**De Klerck** Between Set and Bool: Categories of Aristotelian diagrams	**Mohamed** Chase-Sweedler Galois theory in additive monoidal categories

	Saturday
	Divadlo Bolka Polívky
9:00	**Lemay** Differential Graded Algebras in Differential Categories
10:00	**Kudzman-Blais** Linearly Distributive Fox Theorem
10:30	Coffee
11:00	**Arkor** The three-dimensional structures formed by monoidal categories, bicategories, double categories, etc.
11:30	**Perrone** Categories of relations which compose independently
12:00	**Leinster** The magnitude of a presheaf

Invited talks

TBA

C. Barwick

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Abstract.
TBA

TBA

M.M. Clementino

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Abstract.
TBA

Well-pointed endofunctors on $(\infty, 1)$-categories

S. Henry

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Abstract. (based on a joint work with Mathieu Anel). Well-pointed endofunctors are a tool developed by Kelly in the 80s to formalize most of the "transfinite iterative construction" we encounter in category theory. That is, construction of new objects in a category by repeating a given process an infinite number of times, until it hopefully converges. This includes, for example, colimits in the category of algebras for a monad, free algebras over endofunctors and pointed endofunctors, free monoids, formally inverting an element in a commutative monoid, etc...

In this talk I will present how this theory generalizes to the setting of $(\infty, 1)$-categories, though most of the interesting phenomena are happening at the level of 2-categories, so no knowledge of ∞-category theory is strictly required.

Most of the time, generalizing a result from 1-category theory to $(\infty, 1)$-category theory is relatively straightforward: Once the 1-category theoretic result is presented in a sufficiently nice way, we can just translate it to the ∞-categorical context, to obtain a formally very similar result, by just replacing the basic category theory results used in the proof with their higher categorical analogues. In the rare case where one of the basic results hasn't been established for higher categories yet, we need to establish it, generally using a model-dependent argument.

This talk is about a case where things did not work like this at all: the theory of well-pointed endofunctors on higher categories turn out to look quite different from its 1-categorical counterpart: In the higher categorical setting of the theory, there is an additional obstruction to the convergence of the iteration that appears at the level of 2-cells, and is relatedto braid groups.

Differential Graded Algebras in Differential Categories

Jean-Simon Pacaud Lemay

Abstract.
Differential categories [1] provide a categorical framework for the algebraic foundations of differentiation and also provide the categorical semantics of Differential Linear Logic [2]. There are many interesting examples of differential categories that are based on differentiating various kinds of interesting functions, such as polynomials, power series, smooth functions, etc. Differential categories have been quite successful in formalizing various important concepts related to differentiation such as derivations and Kähler differentials [3], differential algebras [4], de Rham cohomology [6], antiderivatives [5], etc. Following this line of work, in this talk, I will explain how to formalize differential graded algebras in a differential category. I will first provide a friendly introduction to the world of differential categories, leading us to the definition of differential graded algebras relative to a differential category. In models based on polynomial differentiation, we recover the usual commutative differential graded algebras, while in models based on differentiation smooth functions, we recover differential graded $\mathcal{C}^{\infty}$-rings [7]. To further justify our definition, we will explain how the monad of a differential category can be lifted to its category of chain complexes, and how the algebras of the lifted monad correspond precisely to differential algebras of the base category, with the free algebras given by the de Rham complexes. Moreover, it turns out that the category of chain of complexes of a differential category is again a differential category, pointing us towards a possible motivating example of differential dg-categories. This talk is based on joint work with Chiara Sava.

References

[1] Blute, R. F., Cockett, J. R. B., & Seely, R. A. (2006). *Differential categories*. Mathematical structures in computer science, 16(6), 1049-1083.

[2] Ehrhard, T. (2018). *An introduction to differential linear logic: proof-nets, models and antiderivatives*. Mathematical Structures in Computer Science, 28(7), 995-1060.

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Categories or Spaces?
Categorical Concepts in Noncommutative Geometry

W. Lowen

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Abstract.

Ever since the Grothendieck school revolutionized algebraic geometry in the 1960s, categories of sheaves have been central to the subject. Following the seminal work by Artin, Tate and Van den Bergh [1], noncommutative algebraic geometry has taken their prominence to the next level by conceiving noncommutative spaces as categories with certain algebraic generators that can be interpreted as coordinates [4] [3]. That the resulting structures are inherently geometrical is exemplified by their occurrence *in families*, that is, depending on certain parameters that can themselves be organized into so-called moduli spaces. This key feature more generally underpins Kontsevich's Homological Mirror Symmetry Conjecture which revolves around a certain exchange of associated categories for mirror manifolds [8].

In the present talk, we will discuss recent attempts to set up a corresponding *deformation theory* for categorical structures modeling spaces, thus addressing infinitesimal and formal parameters. The prime focus will be on the most relevant categorical tools in play.

We will start by reviewing the basic ideas of algebraic deformation theory in the prototypical example of algebra deformations in relation to affine schemes, as well as an extension to certain projective schemes which borrows ideas from topos theory [12] [16] [11].

Next, we will turn to an approach to deformations of more general schemes via prestacks [15]. More precisely, we will explain how a novel type of "box operadic" structure closely related to the notion of a virtual double category [10] [6] can be used to endow the deformation complex of a prestack with the desired L_∞-structure [7].

Since [8], mirror symmetry has not only been demonstrated for certain noncommutative deformations of classical schemes [2]; it also incorporates "spaces" of a rather different nature, like singularity categories which can be obtained as quotients of two different derived categories associated to a single singular scheme [17]. However, algebraic deformation theory of such categories is notoriously difficult if one uses the classical models of differential graded or A_∞-categories [9].

In this light, we will present the novel conjectural model of *quasi-categories in modules* for this type of categories [13], [14], which is inspired both by Joyal's quasi-categories and by Leinster's up-to-homotopy monoids [10]. We will show that this model is indeed amenable to algebraic deformation theory, and we will sketch some future prospects for the resulting theory [5].

References

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Double categorical equivalences

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Abstract.

Double categories are a flexible 2-dimensional setting that allows us to encode two types of morphisms between objects, as well as a notion of higher cells. Surprisingly, unlike most categorical structures, there is no canonical notion of "equivalence of double categories", as it seems that every possible definition requires us to make a choice. In this talk we will illustrate the issue that arises when defining double-categorical equivalences. Then, we will show how we can use homotopy theory to give a decisive answer as to who the "canonical double categorical equivalences" could be: we give strong evidence towards the claim that these should be the *gregarious equivalences* introduced by Campbell [1].

In the process, we will show how to construct a plethora of model structures on double categories whose homotopy theories encode different 2-dimensional structures. More precisely, we will present an efficient and user-friendly method for constructing any combinatorial model structure on the category of double categories whose trivial fibrations are the "canonical" ones: the double functors which are surjective on objects, full on both horizontal and vertical morphisms, and fully faithful on squares.

This talk will be based on work in preparation joint with Lyne Moser and Paula Verdugo [2].

References

[1] Alexander Campbell, *The folk model structure for double categories*, Seminar talk, `http://web.science.mq.edu.au/groups/coact/seminar/cgi-bin/abstract.cgi?talkid=1616`.

[2] Lyne Moser, Maru Sarazola and Paula Verdugo, Double categorical equivalences, in preparation.

Contributed talks

$(\infty, 2)$-Topoi and descent

Fernando Abellán

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Abstract.
The goal of this talk is to introduce the notion of a Grothendieck $(\infty, 2)$-topos as a presentable $(\infty, 2)$-category satisfying a categorified version of the descent axiom for $(\infty, 1)$-topoi of Rezk-Lurie, which we call fibrational descent. As the name indicates, fibrational descent axiomatizes the structure of internal fibrations in an $(\infty, 2)$-category and it is closely related to the straightening-unstraightening equivalence of Grothendieck-Lurie. After presenting the main definition, I will give an overview of several different ways of characterising $(\infty, 2)$-topoi, which includes a 2-dimensional version of Giraud's theorem and categorified Lawvere-Tierney axioms. Moreover, I will show how the theory of internal categories in an $(\infty, 1)$-topos (as develop by Martini and Wolf) can be embedded into our formalism as $(\infty, 1)$-localic $(\infty, 2)$-topoi. If time permits, I will explain how to construct a version of the Yoneda embedding in any $(\infty, 2)$-topos and a theory of partially lax Kan extensions.

References

[1] Fernando Abellán, Louis Martini, $(\infty, 2)$-*Topoi and descent*, arXiv: 2410.02014

Strongly Finitary Metric Monads are Too Strong

J. Adámek

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Abstract.
For finitary algebras in the cartesian closed categories of sets, posets, cpos, or ultrametric spaces, varieties correspond precisely to the Eilenberg-Moore categories of strongly finitary monads [1, 2]. Several authors (e.g. Rosický, Lucyshyn-Wright, and Parker) have recently asked whether the same holds for **Met**, the category of (extended) metric spaces. It is symmetric monoidal closed, but not cartesian closed. Finitary algebras in this category play an impotant role in computer science, where they are called *quantitative algebras* [3]. For equational presentations one uses *quantitative equations* $t =_\varepsilon t'$ between terms, where $\varepsilon \geq 0$ is a rational number. An algebra satisfies this equation iff every interpretation of variables of t and t' yields elements of distance at most ε.

The answer to the question mentioned above is negative:

Example. The monad for the variety of algebras on binary operations $+$ and $\star$, presented by the quantitative equation $x + y =_1 x \star y$, is not strongly finitary.

Let us denote by $\mathbf{Mnd_p}$ the category of enriched monads on **Met** that are *prefinitary*: they preserve collectively surjective cocones of directed diagrams. The category $\mathbf{Mnd_p}$ is enriched by taking the distance of parallel monad morphisms as the supremum of the distances of their components.

Theorem. *A monad on* **Met** *corresponds to a variety of quantitative algebras iff it is a weighted colimit of strongly finitary monads in* $\mathbf{Mnd_p}$.

A functor between varieties V and W is *concrete* if it commutes with the forgetful funtors of V and W (to **Met**) on the nose.

Corollary. *The following ordinary categories are dually equivalent:*
(1) The category of varieties of quantitative algebras and concrete functors.
(2) The closure of strongly finitary monads under weighted colimits in $\mathbf{Mnd_p}$.

References

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Fundamental Properties of Monads in Double Categories

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Abstract.
It is well established that monoidal categories provide an appropriate context in which internal monoids can be considered. On a higher level, the notion of (pseudo) double category, originally introduced by Ehresmann, can be viewed as a generalization of monoidal categories in which there is accordingly a natural notion of a monad [6, 3].

In this work, we establish various fundamental properties of the category $\mathsf{Mnd}(\mathbb{D})$ of monads and monad maps in a fibrant double category $\mathbb{D}$. For many particular choices of $\mathbb{D}$, the category of monads is a familiar category of interest. For example, if $\mathbb{D}$ is the double category $\mathbb{S}\mathsf{pan}(\mathcal{C})$ of spans in a category $\mathcal{C}$, then $\mathsf{Mnd}(\mathbb{D}) = \mathsf{Cat}(\mathcal{C})$, the category of internal categories in $\mathcal{C}$. Similarly, if $\mathbb{D}$ is the double category $\mathcal{V}$-$\mathbb{M}\mathsf{at}$ of matrices over a monoidal category $\mathcal{V}$, then $\mathsf{Mnd}(\mathbb{D}) = \mathcal{V}$-$\mathsf{Cat}$, the category of enriched categories in $\mathcal{V}$.

On one hand, we consider the existence of limits in $\mathsf{Mnd}(\mathbb{D})$ and their relationship to limits of the same type in categories related to $\mathbb{D}$, such as the category of objects, of arrows or the category of endomorphisms $\mathsf{End}(\mathbb{D})$. In this regard, we rely on a notion of *parallel (co)limits* in a double category, explicitly introduced in [1] – which is not a double-categorical (co)limit, but of a simpler flavor which is of practical use in the fibrant setting.

On the other hand, we investigate the existence of free monads and the monadicity of $\mathsf{Mnd}(\mathbb{D})$ over $\mathsf{End}(\mathbb{D})$. The results here can be seen as a generalization of those in [2], which deals with the case of $\mathcal{W}$-Cat where $\mathcal{W}$ is a bicategory. In the end of this line of ideas, we also approach the question of local presentability of $\mathsf{Mnd}(\mathbb{D})$, in an attempt to generalize known results from the monoidal case [5] and to recover cases of interest like $\mathcal{V}$-Cat [4].

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Open power-objects in categories of algebras

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Abstract. Categories of algebras are seldom toposes, because they usually do not have power-objects. But can they still admit weaker forms of these power-objects? We will see that this is the case for the algebras of monads whose endofunctors and multiplications are *nearly cartesian*.

In a regular category, define a square to be a *near pullback* [1] when its universal morphism into the corresponding pullback is a regular epimorphism. By [1, Theorem 6] (aggregating older results from Carboni *et al.* and Schubert), a functor $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ between regular categories has a monotone extension $\mathbf{Rel}(\mathsf{F}) : \mathbf{Rel}(\mathsf{C}) \to \mathbf{Rel}(\mathsf{D})$ if and only if F is *nearly cartesian*, i.e. preserves near pullbacks; and a natural transformation $\alpha : \mathsf{F} \Rightarrow \mathsf{G}$ between such functors extends to a natural transformation $\mathbf{Rel}(\mathsf{F}) \Rightarrow \mathbf{Rel}(\mathsf{G})$ if and only if α is *nearly cartesian*, i.e. has near pullbacks for its naturality squares.

In particular, as noticed in [2], a monad T on **Set** whose endofunctor and multiplication are nearly cartesian admits a weak distributive law over $\mathbf{P}$ and $\mathbf{P}$ then has a weak lifting $\overline{\mathbf{P}}$ in $\mathbf{EM}(\mathsf{T})$. This was later generalized to the power-object monad of any elementary topos in [1]. Examples of such weakly lifted powerset monads include the Vietoris monad in compact Hausdorff spaces, the convex subset monad in barycentric algebras, the monad of subsets closed under non-empty joins in complete join semi-lattices, and the powerset monad in (commutative) monoids or actions thereof.

In this talk, based on [3], we describe the Kleisli categories of these weakly lifted powerset monads as subcategories of relations. Call a morphism of algebras $f : (R, r) \to (X, x)$ *open* (the name comes from [4, Definition 3.1.1]) when the square $f \circ r = x \circ \mathsf{T}f$ – that makes f into a morphism of algebras – is a near pullback.

Theorem 1. *In* $\mathbf{EM}(\mathsf{T})$*, Kleisli-morphisms* $(X, x) \to \overline{\mathbf{P}}(Y, y)$ *are in one-to-one correspondance with* $\mathbf{EM}(\mathsf{T})$*-relations* $(X, x) \xleftarrow{f} (R, r) \xrightarrow{g} (Y, y)$ *such that* f *is open.*

The main use of this is to characterize monotone extensions to $\mathbf{Kl}(\overline{\mathbf{P}})$ in the spirit of [1, Theorem 6], and thus the existence of monotone weak distributive laws over $\overline{\mathbf{P}}$ in categories of algebras, thereby iterating this construction of weak liftings of powerset monads.

In this talk we will also focus on what this means in terms of elementary-topos-like properties: it entails the existence of a classifier of *open* subobjects, and more generally of an *open* power-object, classifying those relations that are in $\mathbf{Kl}(\overline{\mathbf{P}})$. We will give numerous examples.

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The three-dimensional structures formed by monoidal categories, bicategories, double categories, etc.

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Abstract.

Conventional wisdom in category theory suggests that weak n-categories assemble into weak $(n+1)$-categories. Conventional wisdom in architecture suggests that cubes assemble into three-dimensional structures more easily than spheres. It therefore seems only natural to conclude that weak double categories ought to assemble into a weak triple category. To the chagrin of category theorists and architects alike, this is not the case.

Deciding that composition is not such an important aspect of category theory after all, we investigate the three-dimensional structure that double categories do form, and are led inexorably to the concept of a *virtual triple category* [1]. In this talk, I will explain how weak double categories (and, more generally, virtual double categories) assemble into a virtual triple category. Far from being of interest only to double category theorists, I will show that, by restricting our consideration to various virtual triple subcategories – such as those spanned by monoidal categories, bicategories, and multicategories – we reveal the three-dimensionality inherent to these familiar structures, thereby shedding light on phenomena in two-dimensional and enriched category theory [2, 3, 4, 5, 6, 7].

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Toward the Effective ∞-Topos

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Abstract.

As an elementary topos, Hyland's effective topos $\mathcal{E}ff$ models extensional Martin-Löf type theory with an impredicative universe of propositions [1]. But it also contains another impredicative universe of so-called "modest sets" which is *not* a poset [2]—a fascinating combination not consistent with classical foundations. In joint work with Frey and Speight [3], we proposed a higher-dimensional version of this model with a *univalent* impredicative universe and used it to give new impredicative encodings of some (higher) inductive types. This model was based on cubical assemblies and exploited the constructive character of the recently introduced cubical Quillen model structures [4, 5]. As was subsequently shown, however, the subtopos of 0-types in this model was not equivalent to $\mathcal{E}ff$, but to a larger realizability topos.

In addition to containing a non-degenerate, impredicative, univalent universe, the elementary ∞-topos $\mathcal{E}ff_\infty$ should include $\mathcal{E}ff$ as its subtopos of 0-types. This will provide an example of a non-Grothendieck elementary ∞-topos. As a candidate for $\mathcal{E}ff_\infty$ we here propose an ∞-category of *coherent stacks* over the regular category of assemblies. We show that $\mathcal{E}ff_\infty$ is locally cartesian closed as an ∞-category and that its subcategory of 0-types is indeed $\mathcal{E}ff$.

This is joint work with Mathieu Anel and Reid Barton, building on prior joint work with Jacopo Emmenegger and Pino Rosolini [6].

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Lax Monoidal Structures from Monoidal Structures

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Abstract.
We show that for any full subcategory $\mathcal{C}$ of a (unbiased) monoidal category $\mathcal{D}$, the presheaf category $\mathsf{Set}^{\mathcal{C}^{op}}$ inherits a (unbiased) lax monoidal structure. Our proof, which uses [1], establishes a result of independent interest: that $\mathcal{C}$ inherits a lax promonoidal structure [2]. We then use a generalization of Day convolution to transport this structure into the presheaf category. We discuss examples of lax monoidal structures that arise this way: the cartesian product on simplicial sets [3], the Boardman-Vogt tensor product on dendroidal sets [7], and the Gray tensor product on Θ_2-sets [6]. The first of these is monoidal while the latter two are lax. Future work will study conditions under which the lax monoidal structure constructed via this method are homotopical in the sense of [4], which can be thought of as presenting a monoidal structure in the ∞-categorical sense [5].

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Naturality in Weak ω-categories
(Extended Abstract for arxiv:2501.11620)

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Abstract. Coherence data in a weak higher category, such as associators and unitors, are expected to be natural in their arguments. We show that this is the case of globular weak ω-categories in the sense of Leinster [5]. Explicitly, we produce fillers of naturality squares for every operation in the theory of ω-categories. An application of this work is the construction of cones and cylinders over globular pasting diagrams, as well as their binary compositions, analogous to the work of Lanari [4] for ω-groupoids. Similar to the strict case [3], we expect this to be a step towards a model structure on the category of weak ω-categories. We have implemented our construction in the proof assistant CATT [2] for ω-categories, allowing us to export our proofs to HOTT [1].

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Properties of coalgebraic models of a Lawvere theory

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Abstract.

Objects in the categories of bialgebras, Hopf algebras or Hopf braces have in common an underlying coalgebra structure. When this structure is cocommutative, they can be interpreted as categories of product-preserving functors from a Lawvere theory to the category Coalg of (cocommutative) coalgebras, analogously to how an algebraic theory defines a functor from a Lawvere theory to Set. The idea of passing from models in Set to models in Coalg was considered in [1] where a "linearization process" was used to study some non associative algebraic structures. This process associates a classical term in the algebraic theory with a linearized one, which is a homomorphism of coalgebras, thereby describing, for instance, how group identities transfer to Hopf algebra identities.

Coalgebraic models of a Lawvere theory seem to have some properties that are similar to those of algebraic varieties. We mention, among others, the construction of colimits in the category $\mathsf{Hopf}_{\mathsf{coc}}$ of Hopf algebras presented in [2], that of a free functor from Coalg to $\mathsf{Hopf}_{\mathsf{coc}}$ [3] and recent similar results on the category HBR of Hopf braces —which serve as coalgebraic models of the Lawvere theory of skew braces— obtained in [4]. We describe a general way to construct colimits and free functors between categories of coalgebraic models of Lawvere theories.

In light of the semi-abelian nature of $\mathsf{Hopf}_{\mathsf{coc}}$ [5] and HBR [6], it is natural to investigate protomodularity, regularity or semi-abelianness in these categories more broadly. In fact, it turns out that, under suitable hypotesis, if a category of coalgebraic models admits a forgetful functor to $\mathsf{Hopf}_{\mathsf{coc}}$, it is semi-abelian. This result allows us to construct new examples of semi-abelian categories. In particular, considering the Lawvere theories of radical rings and of digroups, we take their models in coalgebras defining the categories $\mathsf{HRadRng}$ of *Hopf radical rings* and HDGrp of *Hopf digroups*. Hence, the chain of functors $\mathsf{RadRng} \hookrightarrow \mathsf{SKB} \hookrightarrow \mathsf{DiGrp} \to \mathsf{Grp}$, becomes a chain of arrows between categories of coalgebraic models

$$\mathsf{HRadRng} \lhook\joinrel\longrightarrow \mathsf{HSKB} \lhook\joinrel\longrightarrow \mathsf{HDiGrp} \longrightarrow \mathsf{Hopf}_{\mathsf{coc}}$$

where all the involved categories turn out to be semiabelian. Additionally, all functors in this chain admit left adjoints and the first two inclusions determine two Birkhoff subcategories of HDiGrp.

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Lax idempotent monads in homotopy theory

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Abstract.
Lax idempotent monads, also known as Kock–Zöberlein monads, are monads T on a 2-category such that the unit $\eta_T \colon T \Rightarrow TT$ is a fully faithful right adjoint of the multiplication $\mu \colon TT \Rightarrow T$. A typical example is, for a fixed class of colimits, the functor $T \colon \mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}$ that sends a category to its free cocompletion with respect to these colimits.

In this talk, I will describe an extension of the theory of lax idempotent monads to the setting of $(\infty, 2)$-categories. Many facts about lax idempotent monads on 2-categories have analogues for lax idempotent monads on $(\infty, 2)$-categories—in particular, they can be constructed from very minimal data in such a way that all higher coherences "come for free". This is an important feature in $(\infty, 2)$-category theory, since it is generally impossible to construct such coherences "by hand".

What makes this extension desirable is that many important categories in homotopy theory can be described as categories of (co)algebras over an (op)lax idempotent (co)monad. Examples include the $(\infty, 2)$-category of cocomplete $(\infty, 1)$-categories, the $(\infty, 2)$-category of ∞-operads and the $(\infty, 2)$-category of compactly assembled categories, the latter of which plays a crucial role in Efimov's ground-breaking discovery of continuous K-theory. I will discuss some of these examples in detail and show that several important properties of these $(\infty, 2)$-categories are immediate consequences of the theory of lax idempotent monads.

Weights for oplax colimits

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Abstract.

In 2-categories one can consider a notion of colimit whose universal property is expressed in terms of oplax transformations, rather than 2-natural transformations. These are called *oplax colimits*. Examples include the coKleisli category of a comonad and the Grothendieck construction for functors into Cat. By applying the theory of oplax-morphism classifiers for 2-monad algebras, one can express oplax colimits as ordinary weighted colimits of 2-categories for a particular class of weights. This class of weights bears similarities to the class of *PIE weights* — those generated by products, inserters and equifiers — such that some results and perspectives from the analysis of PIE weights in [1], [2] and [3] can be adapted to these "oplax weights".

The talk will provide a summary of the work begun during my PhD with Richard Garner toward describing this class of oplax weights and some related sub-classes. This work includes a presentation of oplax weights as precisely the coalgebras for oplax-transformation classifiers on presheaf categories, as well as a characterisation of the discrete 2-fibrations which arise as categories of elements for these weights. We will also observe that the free completion of a 2-category under this class of colimits admits a rather natural description by a 2-category of "families", containing the coKleisli completion as a sub-completion.

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Compactological Spaces: Constructing condensed mathematics from classical topology

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Abstract.

The theory of condensed mathematics was introduced by Peter Scholze and Dustin Clausen in a sequence of lecture notes ([1, 2]) from 2019 onwards, in part to introduce and work with an abelian replacement for the famously not-so-well-behaved categories of topological abelian groups or more generally topological modules. Among many other concepts [1, 2] introduce quasiseperated condensed sets which serve as a class of non-degenerate and topologically seperated condensed sets. Scholze and Clausen remark that quasiseperated condensed sets correspond to compactological spaces, as introduced by Lucien Waelbroeck from 1967 onwards ([3, 4]) using the language of point-set topology.

We make this relationship explicit and demonstrate that and how it is canonical. Moreover, we prove that condensed sets form the universal ex/reg completion of the regular category of compactological spaces to a Barr-exact category (and Grothendieck topos). This relation carries over to compactological algebraic objects where the description of the completion simplifies drastically. We conclude by demonstrating that condensed modules are given, up to equivalence, by "formal fractions" of their compactological counterpart.

Our investigation, based on a forthcoming paper, does not only respond to the question of the relationship between quasiseperated and all condensed objects, it also permits an accessible description of condensed objects for those without prior contact with algebraic geometry, sheaves or topoi.

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Building pretorsion theories from torsion theories

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Abstract.

Pretorsion theories are defined as "non-pointed torsion theories", where the zero object and the zero morphisms are replaced by a class of "trivial" objects and a suitable ideal of morphisms respectively. Thus, the notion of pretorsion theory can be defined in any arbitrary category $\mathcal{C}$, starting from a pair $(\mathcal{T}, \mathcal{F})$ of full replete subcategories of $\mathcal{C}$ where $\mathcal{T}$ and $\mathcal{F}$ consist of the classes of "torsion" and "torsion-free" objects, and whose intersection defines the class of "trivial objects".

In this talk, we present two ways of obtaining pretorsion theories starting from torsion theories, so that many new examples of pretorsion theories can be given in pointed categories. After recalling some key background on torsion and pretorsion theories, we shall describe pretorsion theories coming from pairs of torsion theories. Lattices and chains of torsion theories are widely studied topics and they are the perfect framework for applying our result. Then, we shall show how to obtain pretorsion theories "extending" a torsion theory with a Serre subcategory. We shall discuss some applications in representation theory and in the framework of recollements of abelian categories.

This talk is based on a joint work with Francesca Fedele (University of Leeds).

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2-classifiers for 2-algebras

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Abstract. In [2], Paré investigates the Yoneda theory of double categories, giving a construction of *lax* double functors $D(d, -) : D \to \mathbb{S}\mathbf{pan}(\mathbf{Set})$ represented by objects of double categories. The double categories of elements $\mathbf{El}(F)$ of lax functors $F : D \to \mathbb{S}\mathbf{pan}(\mathbf{Set})$ are discretely fibered over D by a *strict* double functor $\mathbf{El}(F) \to D$; as shown in [3]. This gives an equivalence between the categories of *lax* functors $D \to \mathbb{S}\mathbf{pan}(\mathbf{Set})$ and the category of *strict* discrete fibrations over D.

Motivated by applications in categorical systems theory, we generalize this equivalence to general 2-algebraic theories, showing that discrete opfibration classifiers with suitable 2-algebraic structure classify *strict* discrete opfibrations in the 2-category of algebras and lax morphisms.

More precisely, let $\mathcal{K}$ be an enhanced 2-category equipped with a *good 2-classifier* $1 \xrightarrow{\tau} \Omega$ in the sense of Mesiti [1], meaning it classifies tight discrete opfibrations. We ask: when can we lift such a structure to the enhanced 2-category of (strict) 2-algebras (with tight strict morphisms as tights and lax morphisms as looses) of an enhanced 2-monad T on $\mathcal{K}$?

We single out the following condition:

Definition 1. *An* Ω***-structure*** *on* T *is a* T*-algebra structure* $\omega : T\Omega \to \Omega$ *which classifies* Tu*, where* $u : 1/\Omega \to \Omega$ *is the free (tight) discrete opfibration.*

The theorem we prove is the following:

Theorem 1. *Let* T *be an enhanced 2-monad which moreover (1) preserves pullbacks of tight discrete opfibrations and (2) has tight-cartesian*[2] *unit and multiplication. Then an* Ω*-structure on* T *induces an enhanced 2-classifier in the enhanced 2-category of* T*-algebras and lax morphisms.*

Crucially, the construction can be easily iterated since $\mathbf{Alg}_l(\mathcal{K})$ is again an enhanced 2-category with a good 2-classifier. Indeed, our main applications are: (1) for $\mathcal{K} = \mathbf{Cat}$ and $T =$ free SMC, (2) $\mathcal{K} = \mathbf{Cat}^{\cdot \rightrightarrows \cdot}$ and $T =$ free double category. Since the free SMC construction lifts to $\mathbf{Cat}^{\cdot \rightrightarrows \cdot}$, we see how by applying Theorem 1 twice we can lift $1 \xrightarrow{\{*\}} \mathbf{Set}$ from $\mathbf{Cat}$ to double categories (recovering [2]) to symmetric monoidal double categories.

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Structures for the category of torsion theories

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Abstract.

Torsion theories were introduced in [3] to generalize the concept of torsion and torsion-free abelian groups to the context of abelian categories. Since then, a rich theory has been developed, even in more general categorical contexts. Despite this, little is known in the literature about the category of torsion theories and appropriate functors between them. We contribute to fill this gap by studying various structures for the 2-category of torsion theories.

We construct the cofree torsion theory on a pointed category. Explicitly, this is given by the category of short exact sequences in the starting pointed category. We then extend this construction to prove a comonadicity result for torsion theories. More precisely, the 2-category of torsion theories is comonadic over the 2-category of pointed categories with functors that preserve the zero object and short exact sequences. We can thus apply such result to study (2-dimensional) colimits in the 2-category of torsion theories.

We then prove that the 2-category of torsion theories has all products, calculated in the 2-category of pointed categories. This makes us wonder whether torsion theories are also monadic over the 2-category of pointed categories. While such result does not seem to hold in full generality, we prove a partial monadicity result. More precisely, we show that those torsion theories $(\mathbb{C}, \mathcal{T}, \mathcal{F})$ for which the canonical functor $\mathbb{C} \to \mathcal{T} \times \mathcal{F}$ is an equivalence of categories are monadic over the 2-category of pointed categories with functors that preserve the zero object. The involved 2-monad sends a pointed category $\mathbb{C}$ to the product $\mathbb{C} \times \mathbb{C}$. So, interestingly, this 2-monad is a 2-dimensional analogue of the monad on **Set** whose algebras are *rectangular bands* in semigroup theory [4], i.e. idempotent semigroups satisfying $xyz = xz$. We thus see the special torsion theories described above as internal rectangular bands in the 2-category of pointed categories, and we call them *rectangular torsion theories*. We prove that rectangular torsion theories are precisely the ones equivalent to products in the 2-category of torsion theories of form $(\mathcal{T}, \mathcal{T}, \mathbf{0}) \times (\mathcal{F}, \mathbf{0}, \mathcal{F})$, where $\mathbf{0}$ is the full subcategory of the base category consisting of all zero objects.

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Gray products of diagrammatic (∞, n)-categories

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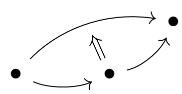
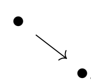
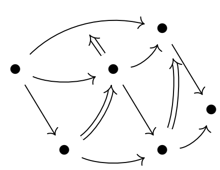
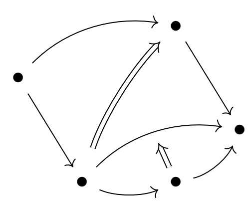
Abstract.

The Gray tensor product serves as a directed refinement of the cartesian product in higher categorical settings and is particularly useful to model all sorts of (co)lax constructions. In this talk, we present diagrammatic sets, home of the diagrammatic model of (∞, n)-categories [1], and their interaction with the Gray tensor product. Diagrammatic sets are sheaves over a category of combinatorial pasting diagrams, known as *regular directed complexes*, and main subject of Hadzihasanovic's recent monograph [2], where it is shown that regular directed complexes are monoidal with respect to the Gray product $- \otimes -$. Pictorially, we can represent the Gray product of the 2-cell and the 1-cell

 ,

respectively, as a "right cylinder" 3-cell with input and output 2-boundaries

 and  respectively.

After presenting some of the features of this construction, like its interaction with the pasting of pasting diagrams, or its distributivity under duals, we extend the Gray product to diagrammatic sets via the theory of Day convolution, and study some of its notable properties, following the work in progress [3]. For one, distinguishing the Gray tensor product from the cartesian product, we consider its interaction with higher invertible *round diagrams* (read *cells*), known as equivalences: *if u is an equivalence in X **or** v is an equivalence in Y, then $u \otimes v$ is an equivalence in $X \otimes Y$.* For another, we demonstrate that the Gray product preserves ω-equivalences, that is, morphisms $f : X \to Y$ of diagrammatic sets inducing an essential surjection hom-wise $f : X(u, v) \to Y(f(u), f(v))$ for all parallel pairs u, v. This result is of homotopical importance: ω-equivalences are precisely the weak equivalences of diagrammatic (∞, n)-categories for the model structure described in [1].

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Closed categories, pro-operads and Goodwillie calculus

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Abstract.

Eilenberg and Kelly [EK66] introduced the notion of a closed, not necessarily monoidal, structure on a category. We show that the category of pro-objects in a closed category $\mathcal{V}$ inherits a closed structure, which is not typically monoidal even when $\mathcal{V}$ is. Our strategy is to embed $\mathrm{Pro}(\mathcal{V})$ into the opposite of the category of $\mathcal{V}$-enriched endofunctors on $\mathcal{V}$.

Our motivation is the case where $\mathcal{V}$ is the category of symmetric sequences with (right-)closed monoidal structure given by the composition product, and our application is to Goodwillie's functor calculus. Using previous work with Arone [AC15, AC16], we show that the Goodwillie-Taylor towers of spectrum-valued functors on a pointed ∞-category C are classified by right modules over a 'pro-operad', that is, a monoid in the closed structure on pro-symmetric sequences. That pro-operad is an example of an 'endomorphism pro-operad' constructed from a certain inverse sequence of functors from C to spectra.

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Homotopical recognition of diagram categories

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Abstract.

Building on work of Marta Bunge in the one-categorical case, we characterize when a given model category is Quillen equivalent to a presheaf category with the projective model structure. This involves introducing a notion of *homotopy atoms*, generalizing the orbits of Dwyer and Kan, [2].

As an application, we give a classification of polynomial functors (in the sense of Goodwillie calculus, [3]) from finite pointed simplicial sets to spectra, and compare it to the previous work by Arone and Ching, [1]. Since the answer is a category of simplicial functors taking values in the simplicial category of spectra, we have to adapt the concept of homotopy atoms and present it as a Kelly product of two categories satisfying part of the properties.

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A new framework for limits in double categories

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Abstract. Robert Paré introduced the notion of limit in a double category at the International Category Theory Conference in 1989, showing that weighted limits in a 2-category arise as a special case. The subsequent paper with Marco Grandis [1] presents an elegant theory of double limits, and proves that the limit of any double functor $F\colon \mathbb{J} \to \mathbb{D}$ exists if and only if the double category $\mathbb{D}$ has small double products, double equalisers, and *tabulators*. Many double categories admit all limits and colimits in this sense, including the double category $\mathbb{S}\mathrm{pan}$ of sets, functions, and spans, and the double category $\mathbb{D}\mathrm{ist}$ of categories, functors, and distributors (a.k.a. profunctors).

Among the most fundamental concepts in double category theory are those of *companion* and *conjoint*, together with closely related notions of *restriction* and *corestriction*. A double category has companions and conjoints if and only if it has restrictions if and only if it has corestrictions—in this case, it is called an *equipment* or *framed bicategory* [2]. Another important concept is a *local (co)limit* in a double category [3]. For example, local coequalisers are required to construct the double category $\mathbb{M}\mathrm{od}(\mathbb{D})$ of monads, monad morphisms, and bimodules in a double category $\mathbb{D}$. Despite the central role they play, neither restrictions (thus companions and conjoints) nor local limits are captured by the limit of a double functor, which is defined as an *object* rather than a *loose morphism* in the double category.

In this talk, we introduce a new framework for limits in double categories which extends the work of Grandis and Paré, and captures companions, conjoints, restrictions, and local limits as examples. The novel aspect of this framework is to use a *loose distributor* $J\colon \mathbb{S} \nrightarrow \mathbb{T}$ between double categories as the *shape* of a diagram. Our main theorem characterises when a double category admits all limits of diagrams in this new sense; examples include both $\mathbb{S}\mathrm{pan}$ and $\mathbb{D}\mathrm{ist}$.

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A new tool for detecting cofibrant generation

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Abstract.

A class $\mathcal{M}$ of monomorphisms is *cofibrantly generated* if it is the closure of a (small) set of morphisms under pushouts, transfinite compositions, and retracts. If $\mathcal{M}$ is cofibrantly generated then, under certain technical conditions, Quillen's Small Object Argument ensures that $\mathcal{M}$ forms the left part of a Weak Factorization System.

In locally finitely presentable categories, Borceux and Rosický [1] isolate a condition ("effective unions of pure subobjects") that is sufficient, but not necessary, for the class of pure monomorphisms to be cofibrantly generated. We noticed that if their condition is only required to hold "almost everywhere", then it exactly characterizes when pure monos are cofibrantly generated. The notion of "almost everywhere" can be made precise in several ways, most concisely via a certain interpretation of Shelah's *Stationary Logic*. Moreover, the characterization works for any class $\mathcal{M}$ of monomorphisms that is closed under pushouts, transfinite compositions, and retracts; i.e., such a class is cofibrantly generated if and only if one has "almost everywhere" effective unions of $\mathcal{M}$-subobjects.

As an application, we prove:

Theorem: Suppose $\mathcal{M}$ is a monoid, and that the class of pure monomorphisms is cofibrantly generated in the category of $\mathcal{M}$-acts. Then $\mathcal{M}$ has only a set of locally cyclic acts. (an $\mathcal{M}$-act A is locally cyclic if for every $a, b \in A$, the subact $\langle a, b\rangle$ is contained in a cyclic subact of A.)

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Projective crossed modules in semi-abelian categories

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Abstract.
We characterize the projective objects in the category of internal crossed modules [5, 4] within any (Janelidze–Márki–Tholen [6]) semi-abelian category. When this category forms a variety of algebras, the internal crossed modules again constitute a semi-abelian variety, ensuring the existence of free objects, and thus of enough projectives. We show that such a variety is not necessarily Schreier—subojects of free objects are again free—, but does satisfy the so-called Condition (P) [3]—meaning the class of projectives is closed under protosplit subobjects—if and only if the base variety satisfies this condition. As a consequence, the non-additive left chain-derived functors of the connected components functor are well defined (in the sense of [3]) in this context.

The main references of this talk are [2, 1].

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Schemes relative to Actegories

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Abstract.
In [3], Toën and Vaquié give a category-theoretic notion of a scheme over a closed symmetric monoidal category $\mathcal{C}$ which generalizes the usual algebraic geometry over $(Ab, \otimes_{\mathbb{Z}}, \mathbb{Z})$. They consider the category $Comm(\mathcal{C})$ of internal commutative monoids in $\mathcal{C}$ and define a Grothendieck topology J on $Aff_{\mathcal{C}} := Comm(\mathcal{C})^{op}$. To do this, they use actions of objects $c \in Comm(\mathcal{C})$ on objects of $\mathcal{C}$. By choosing $\mathcal{C}$ appropriately, one opens up a number of new geometries by taking $\mathcal{C}$ for instance to be the category of sets, or commutative monoids, or the category of symmetric spectra.

In this talk, we bring together their novel idea and the notion of vertical categorification of monoid actions, popularly known as "Actegories" ([2]), to present a notion of a scheme relative to a $\mathcal{C}$-actegory $\mathcal{M}$ ([1, § 4]). This is motivated by the Microcosm Principle, due to Baez and Dolan, which suggests that Actegories are the right setups for internalizing monoid actions, just as (braided/symmetric) monoidal categories are the right setups to internalize (commutative) monoids.

We start with a datum $(\mathcal{C}, \mathcal{M})$ consisting of a closed symmetric monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, 1)$ and a left $\mathcal{C}$-actegory $\mathcal{M} = (\mathcal{M}, \boxtimes : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M})$ such that the $\mathcal{C}$-action bifunctor $\boxtimes$ is cocontinuous in both variables. We then use actions of objects $c \in Comm(\mathcal{C})$ on objects of $\mathcal{M}$ to define a Grothendieck topology $J_{\mathcal{M}}$ on $Aff_{\mathcal{C}} = Comm(\mathcal{C})^{op}$. In general, $J_{\mathcal{M}}$ may not be subcanonical (unlike the case of [3, Corollary 2.11] where $(\mathcal{M}, \boxtimes) = (\mathcal{C}, \otimes)$). For technical purposes, we only work with those $\mathcal{M}$ for which $J_{\mathcal{M}}$ is subcanonical (and call $\mathcal{M}$ a "subcanonical actegory"). We then define an "$\mathcal{M}$-scheme" as an object of the Grothendieck topos $Sh(Aff_{\mathcal{C}})_{\mathcal{M}}$ (of sheaves on the site $(Aff_{\mathcal{C}}, J_{\mathcal{M}})$) which can be "nicely covered" by representable sheaves. We see that the full subcategory $Sch_{\mathcal{M}}$ of $Sh(Aff_{\mathcal{C}})_{\mathcal{M}}$, consisting of the $\mathcal{M}$-schemes, is closed under pullbacks, coproducts and certain quotients.

We will end with a theorem which investigates the behaviour of our notion of a scheme under base changes along an adjunction in the 2-category $MonCat_{lax}$ and a lax linear functor between actegories. We also use this to generate a plethora of examples of subcanonical actegories.

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Between Set and Bool: Categories of Aristotelian diagrams

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Abstract.
Aristotelian diagrams, such as the square of opposition, have long been used as pedagogical tools in logic and philosophy [3]. With the conception of logical geometry in the early 21st century, they became objects of study in their own right [4]. Formally, a (classical) Aristotelian diagram can be viewed as a couple $(\mathcal{F}, B)$, where $\mathcal{F}$ is a subset of a Boolean algebra B. When visualizing these diagrams, the Aristotelian relations (contradiction, (sub)contrariety and subalternation) that hold between the elements of $\mathcal{F}$ are drawn. Therefore, Aristotelian diagrams exhibit two different levels of structure. The first level is defined solely by the Aristotelian relations, while the second level also cares about the other identities that hold between the elements of $\mathcal{F}$ in B.

To establish solid mathematical foundations for the study of Aristotelian diagrams, recent research in logical geometry has incorporated category theory, by creating categories in which the objects are precisely the Aristotelian diagrams [1]. Among them, the category $\mathbb{D}^{Inc}_{\mathcal{OR}\times\mathcal{IR}}$ looked the most promising to describe the first strucutral level of these diagrams.

In a forthcoming book, we deepen this category-theoretical approach to logical geometry [2]. In this book, we show that $\mathbb{D}^{Inc}_{\mathcal{OR}\times\mathcal{IR}}$ is bicomplete by providing constructions for all its (small) limits and colimits. Additionally, we introduce the category $\mathbb{D}_{\mathcal{B}}$, which captures the second structural level of Aristotelian diagrams, and prove its bicompleteness by providing constructions for all its (small) limits and colimits. The (co)limits in these categories relate closely to those in Set and Bool, and examples of each of them can be found in previous research in logical geometry (which did not yet use the language of category theory). Furthermore, we establish three adjunctions: between Set and $\mathbb{D}^{Inc}_{\mathcal{OR}\times\mathcal{IR}}$, between $\mathbb{D}^{Inc}_{\mathcal{OR}\times\mathcal{IR}}$ and $\mathbb{D}_{\mathcal{B}}$, and between $\mathbb{D}_{\mathcal{B}}$ and Bool. Their composition is proven to be precisely the free/forgetful adjuntion between Set and Bool. These results, though proven, remain unpublished as the book is still in progress. We hope to present them for the first time at CT2025.

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Category Theory within a 2-Category: Internal Enrichment, Presheaf Objects and Convolution Products

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Abstract.

Category theory offers convenient tools (such as Kan extensions and Day convolution) for the description of mathematical structures. However, the 2-category of categories is not general enough to encapsulate certain structures. For instance, it is often needed to place within a $\mathcal{V}$-enriched setting for a closed monoidal category $\mathcal{V}$. More generally, it has been noticed that most of the tools of category theory (equivalences, adjunctions, Kan extensions…) can be generalized to the framework of a 2-category. While $\mathcal{V}$-enriched ends and coends provide explicit formulas for $\mathcal{V}$-natural transformations, Kan extensions, and Day convolution product within the 2-category of $\mathcal{V}$-enriched categories, more general 2-categories lack such formulas. Based on the first chapter of [1], this work is motivated by the example given by the 2-category of symmetric sequences of categories $\mathrm{CAT}^{\mathfrak{S}}$, with the goal of defining the presheaf operad of an operad - notably, the presheaf operad of the face poset of the associahedra.

The purpose is to extend categorical constructions to the framework of a 2-category with enough structure, pushing the internalization process further while aligning with the principles of formal category theory. We proceed to define internal enrichment, from which we obtain an internal definition of ends and coends, notably providing explicit formulas for Kan extensions - hence Day convolution. For this purpose, we define suitable notions on a cartesian closed 2-category $(\Lambda, \times_\Lambda, [\_, \_]_\Lambda)$ so that it sufficiently ressembles CAT. Those notions include an oppositization 2-functor $(\_)^{op} : \Lambda^{op_2} \to \Lambda$, an internally complete and cocomplete closed monoidal object $(\mathcal{S}, \times_\mathcal{S}, [\_, \_]_\mathcal{S})$ in Λ - e.g. SET, $\mathcal{V}$, $\mathrm{SET}^{\mathfrak{S}}$ within CAT, resp. $\mathrm{CAT}_\mathcal{V}$, resp. $\mathrm{CAT}^{\mathfrak{S}}$. We first define relative notions of ends and coends, relatively to a morphism $F : \mathcal{C}^{op} \times_\Lambda \mathcal{C} \to \mathcal{S}$ in Λ. We introduce the notion of an $\mathcal{S}$-enrichment in Λ, which involves a coherent data of morphisms $\mathcal{C}(\_, \_) : \mathcal{C}^{op} \times_\Lambda \mathcal{C} \to S$ in Λ for each object $\mathcal{C}$ of Λ. This notion of coherence is expressed by using a 2-functor (see I.4.3 of [1]), and yields the following isomorphism in $\mathcal{S}$

$$[\mathcal{C}, \mathcal{D}]_\Lambda(F, G) \simeq \int^{\mathcal{C}(\_,\_)} \mathcal{D}(F\_, G\_)$$

for objects $\mathcal{C}, \mathcal{D}$ and $F, G : \mathcal{C} \to \mathcal{D}$. We define ends and coends of morphisms $\mathcal{C}^{op} \times_\Lambda \mathcal{C} \to \mathcal{S}$ as those relative to $\mathcal{C}(\_, \_)$. As a result, we obtain a pointwise expression of the internal left Kan extension

$$\mathrm{Lan}^\mathcal{S} : [\mathcal{C}_2, \mathcal{C}_1]^{op}_\Lambda \to [[\mathcal{C}_2, \mathcal{S}]_\Lambda, [\mathcal{C}_1, \mathcal{S}]_\Lambda]_\Lambda$$

whose value $\mathrm{Lan}^\mathcal{S}_\mu$ on $\mu : \mathcal{C}_2 \to \mathcal{C}_1$ is the internal left adjoint of $[\mu, \mathcal{S}]_\Lambda$, given in $F : \mathcal{C}_2 \to S$ by $\mathrm{Lan}^\mathcal{S}_\mu F \simeq \int_{x:*_\Lambda \to \mathcal{C}_2} \mathcal{C}_1(\mu x, \_) \times_\mathcal{S} Fx$. In addition, we obtain an internal version of Yoneda embedding, which satisfies the corresponding lemma and exhibits the presheaf object $[\mathcal{C}^{op}, \mathcal{S}]_\Lambda$ of any object $\mathcal{C}$ as the free cocomplete completion of $\mathcal{C}$ in Λ.

We eventually consider the case where Λ is further equipped with an additional monoidal structure $(\Lambda, \otimes_\Lambda)$ which is compatible with its former closed monoidal structure. Extending Day's convolution product to this framework, we obtain a monoidal structure on the Yoneda embedding which exhibits the presheaf object of an internal monoid $(\mathcal{C}, \otimes_\mathcal{C})$ as the free cocomplete completion of $\mathcal{C}$ in the 2-category of monoids internal to $(\Lambda, \otimes_\Lambda)$ - provided that $\mathcal{S}$ has an additional internal monoidal structure as well.

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Extending an arithmetic universe by an object

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Abstract.

In a 1999 paper [4], Vickers speculated that arithmetic universes could be used to provide a "base-free" way to study Grothendieck toposes. As he explains in [5], one usually works with an elementary topos $\mathcal{S}$ as a base, and a Grothendieck topos with respect to $\mathcal{S}$ is an elementary topos together with a bounded geometric morphism to $\mathcal{S}$. In this way, the infinitary operations of the Grothendieck topos are parametrized by objects of $\mathcal{S}$. A problem with this approach is that the most natural notion of morphism between elementary toposes is not geometric morphisms, so the category of elementary toposes and geometric morphisms is not as well behaved.

On the other hand, morphisms of arithmetic universes (that is, list-arithmetic pretoposes [2]) are much closer to geometric morphisms. Instead of having all power objects, these categories have parametrized list objects $L(X)$ for every object X; thus, the only infinities available are free algebra constructions.

Vickers' idea led to a research program exploring whether it is possible to develop the theory of Grothendieck toposes using AUs, which has seen some success [3, 5]. For instance, Vickers recently [5] developed an analogue for classifying toposes; specifically, he gave a syntactic construction of the arithmetic universe $\mathbf{AU}\langle\mathbb{T}\rangle$ freely generated by a context $\mathbb{T}$. This extends to a full and faithful 2-functor from contexts to arithmetic universes.

In this talk, I'll examine the special case of extending an arithmetic universe $\mathcal{A}$ by a single indeterminate object X to get $\mathcal{A}[X]$. I'll show that the same description as for Grothendieck toposes holds [1, B3.2.9]: $\mathcal{A}[X]$ is equivalent to the category $\mathrm{CoPsh}(\mathbf{Fin}_{\mathcal{A}})$ of internal copresheaves on the internal category $\mathbf{Fin}_{\mathcal{A}}$ of finite sets in $\mathcal{A}$. Looking forward, one could consider the extension by an arbitrary collection of objects; together with localization, this would give a concrete description of classifying toposes over an arbitrary arithmetic universe, analogous to the case of a base elementary topos.

This work is the result of PhD research under the supervision of Simon Henry and Philip J. Scott, and is based on a conjecture of Simon Henry.

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Call *doctrines* by your name

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Abstract.
What's a *doctrine* to you? Is it a **lax-idempotent pseudomonad**, is it some form of Lawverian **doctrine**, or a type of **classifying topoi** for a specific fragment of predicate logic?

We show that:

(1) a class of geometric morphisms $\mathcal{H}$ can be understood as specifying a fragment of geometric logic. A topos $\mathcal{E}$ is Kan injective with respect to $\mathcal{H}$ when its models have the desired property specified by $\mathcal{H}$.

(2) every such class of morphisms induces a lax-idempotent relative pseeudomonad $\mathcal{H} \mapsto \mathsf{T}^{\mathcal{H}}$ on the 2-category Lex. Every algebra for such a monad has a *classifying topos*, $\mathcal{C}l : \mathsf{Alg}(\mathsf{T}^{\mathcal{H}})^{\mathrm{op}} \to \mathsf{RInj}(\mathcal{H})$. Let's see two extreme cases below, but the real fun is *everything in between*.

- when $\mathcal{H}$ is all geometric morphisms, $\mathsf{T}^{\mathcal{H}}$ is the identity and $\mathcal{C}l$ is computing the classifying topos of a lex theory, i.e. its presheaf topos.
- when $\mathcal{H}$ is empty, $\mathsf{T}^{\mathcal{H}}$ is the presheaf construction and $\mathcal{C}l$ is *tautological* duality between Logoi and Topoi.

(3) Every lax-idempotent pseudomonad T on Lex induces a lax-idempotent pseudomonad $\mathsf{T}^{\mathrm{fbr}}$ on a reasonable 2-category of Lawverian doctrines, so that we have a representation functor,

$$\mathsf{Sub} : \mathsf{Alg}(\mathsf{T}) \to \mathsf{Alg}(\mathsf{T}^{\mathrm{fbr}}).$$

When T is the identity, so is $\mathsf{T}^{\mathrm{fbr}}$. When T is the presheaf construction, $\mathsf{T}^{\mathrm{fbr}}$ is the free locale completion of a dependent meetlattice.

These constructions allow to give definitions (!) of what *is* a (fragment of geometric) logic, and to move between these notions *safely*, recovering several constructions from the classical literature of categorical logic (classifying topos, syntactic category, doctrine of subobjects...).

This talk presents two ongoing collaborations. One with **Lingyuan Ye** (on (1) and (2)), the other with **Joshua Wrigley** and **Jacopo Emmenegger** on (3).

Cartesian monoidality of the cubical Joyal model structure

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Abstract.
Modelling homotopy theory using cubical sets requires a suitable monoidal product, with respect to which homotopies can be defined. For less structured varieties of cubical sets, such as those having only faces, degeneracies and connections, the cartesian product is often difficult to work with; thus it is more common to use the geometric product when modelling homotopy theory using these cubical sets. This has the advantage of an elegant description, as in contrast to the cartesian product, the geometric product of cubes is again a cube, but has the disadvantage that it is not symmetric, leading to two distinct constructions of mapping spaces between cubical sets. In this talk, based on the paper [2], we will discuss a proof that the cubical Joyal model structure on cubical sets with connections (constructed in [1]), which models $(\infty, 1)$-categories, is monoidal with respect to the cartesian product. This is done by means of a comparison with more structured cubical sets having not only connections, but also symmetries and diagonals, which also allows for the construction of a Quillen-equivalent model structure on the latter category. Moreover, this comparison also allows us to obtain a new proof that the geometric product of cubical sets is symmetric up to natural weak equivalence, even in the absence of connections.

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An adjunction between categories of monads

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Abstract

Given a monad T on $\mathscr{A}$ and a functor $G\colon \mathscr{A} \to \mathscr{B}$, if the right Kan extension of GT along G exists, then it has a canonical monad structure; we call this monad the pushforward of T along G and denote it by $G_{\#}T$. If F is left adjoint to G, then $G_{\#}T$ is the familiar monad structure on GTF. If T is the identity monad, then $G_{\#}T$ is the codensity monad of T. Even though it was introduced by Street [2] in the 1970s, the pushforward constructions has not yet been studied in depth. This talk will report on work towards this goal, the details of which can be found in [1].

Under suitable smallness and completeness conditions, $G_{\#}$ becomes a functor $\mathsf{Mnd}(\mathscr{A}) \to \mathsf{Mnd}(\mathscr{B})$. In fact, when G is full and faithful, $G_{\#}$ has a partial left adjoint $G^{\#}$, defined on those monads on $\mathscr{B}$ whose endofunctor restricts to $\mathscr{A}$, i.e. those monads S such that there is an endofunctor S' of $\mathscr{A}$ and an isomorphism $SG \cong GS'$. We denote the corresponding subcategory of $\mathsf{Mnd}(\mathscr{B})$ by $\mathsf{Mnd}(\mathscr{B})^{\mathrm{res}\mathscr{A}}$, giving an adjunction

$$\begin{tikzcd} \mathsf{Mnd}(\mathscr{A}) \arrow[r, shift left=1.5, "G_{\#}"] \arrow[r, phantom, "\top"] & \mathsf{Mnd}(\mathscr{B})^{\mathrm{res}\mathscr{A}} \arrow[l, shift left=1.5, "G^{\#}"] \end{tikzcd} \subseteq \mathsf{Mnd}(\mathscr{B}),$$

with $G_{\#}$ full and faithful and $G^{\#}$ given by restriction.

The conditions which result in this situation are not hard to satisfy. For example, one may take G to be the inclusion $\mathsf{FinSet} \hookrightarrow \mathsf{Set}$, the inclusion $\mathsf{Set}_{\neq\varnothing} \hookrightarrow \mathsf{Set}$ (where the domain is the category of nonempty sets), the discrete-topology functor $\mathsf{Set} \to \mathsf{Top}$, or the forgetful functor $\mathsf{Field} \to \mathsf{CRing}$. The first two will be the guiding examples for this talk.

In the case of $\mathsf{Set}_{\neq\varnothing} \hookrightarrow \mathsf{Set}$, the category on the right-hand side of the adjunction is just $\mathsf{Mnd}(\mathsf{Set})$, and the corresponding reflective subcategory on the left is spanned by those monads all of whose pseudoconstants are induced by actual constants.

In the case of $\mathsf{FinSet} \hookrightarrow \mathsf{Set}$, the right-hand side of the adjunction includes monads such as the identity, the $(-) + E$ monad (for a finite set E), the $M \times (-)$ monad (for a finite monoid M), the powerset monad, the double-powerset monad, the ultrafilter monad, and the filter monad. I will explain what the pushforward of their restriction to FinSet is; for example, starting with the powerset monad, this process produces the theory of continuous lattices. I will also explain why, with two trivial exceptions, pushforwards along $\mathsf{FinSet} \hookrightarrow \mathsf{Set}$ never have rank.

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The monad of factorizations and its decomposition

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Abstract.

Given a short exact sequence $K \xrightarrow{k} X \xrightarrow{q} Q$ in a semi-abelian category $\mathcal{C}$, and a morphism $\varphi\colon K \to L$, one can show [1] that φ can be extended to a morphism

$$\begin{array}{ccccc} K & \xrightarrow{k} & X & \xrightarrow{q} & Q \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle f} & & \| \\ L & \xrightarrow[l]{} & Y & \xrightarrow[q']{} & Q \end{array} \tag{1}$$

of short exact sequences if there exists an action $\xi_L^X\colon X\flat L \to L$ of X on L such that φ is a morphism of X-actions $(K, \chi_K^X) \to (L, \xi_K^X)$ and

$$(\varphi \rtimes X)^* \chi_L^{L\rtimes X} = [k, 1\rangle^* \xi_L^X. \tag{2}$$

In this talk, we give a new interpretation of this result and explore some generalizations and applications. To this end, we consider the category $\mathcal{C}^q$ whose objects are morphisms (1) for a fixed regular epimorphism q, which may be seen as a co-slice of the category of short exact sequences in $\mathcal{C}$. Taking the object L yields a functor $\mathcal{C}^q \to \mathcal{C}$. We show that this functor is monadic.

As particular cases of monads T^q induced by an adjunction of this type, we obtain the monads of internal actions $L \mapsto X\flat L$, as well as the monads of the form $X \mapsto A + X$ for a fixed object A. Furthermore, we show that the functor part of such a monad admits a natural decomposition as a semidirect product. Using this fact, we show that its algebras can similarly be decomposed into an action $\xi_L^X\colon X\flat L \to L$ and a morphism $\varphi\colon K \to L$. We then show that the pairs (ξ_L^X, φ) that give rise to a T^q-algebra are precisely those satisfying the equivariance condition and (2).

We will then explain how such monads can be used to describe higher order semidirect products (as defined by Carrasco and Cegarra [2, 3]) in semi-abelian categories.

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Action representability of internal 2-groupoids

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Abstract. Based on [1], we prove that, in any regular Mal'tsev category $\mathscr{C}$ with coequalizers, the category 2-Grpd($\mathscr{C}$) of internal 2-groupoids is a Birkhoff subcategory of the category $\mathrm{Grpd}^2(\mathscr{C})$ of internal double groupoids, i.e., it is a full reflective subcategory that is also closed under subobjects and quotients. It follows that 2-Grpd($\mathscr{C}$) is a semi-abelian category if $\mathscr{C}$ is so, and we identify sufficient conditions on $\mathscr{C}$ for 2-Grpd($\mathscr{C}$) to be action representable if $\mathscr{C}$ is so. This result applies, for example, to the categories Grp of groups, Lie_R of Lie algebras over a commutative ring R, and $\mathrm{Hopf}_{\mathrm{coc},K}$ of cocommutative Hopf algebras over a field K.

If $\mathscr{C}$ is a Mal'tsev category, i.e., a finitely complete category in which any internal reflexive relation is an equivalence relation, then the inclusion functor from the category Grpd($\mathscr{C}$) of internal groupoids to the category Cat($\mathscr{C}$) of internal categories is an isomorphism of categories, and the inclusion functor from Cat($\mathscr{C}$) to the category RG($\mathscr{C}$) of internal reflexive graphs is full. Examples of Mal'tsev categories include any Mal'tsev variety, i.e., a variety of universal algebras whose theory admits a ternary term p that satisfies the identities $p(x, x, y) = y = p(y, y, x)$, such as the varieties of groups, of rings, of Lie algebras, and of Heyting algebras.

It is known that, if $\mathbb{V}$ is a Mal'tsev variety, then 2-Grpd($\mathbb{V}$) is a subvariety of the Mal'tsev variety $\mathrm{Grpd}^2(\mathbb{V})$. Generalizing this result, we show that 2-Grpd($\mathscr{C}$) is a Birkhoff subcategory of $\mathrm{Grpd}^2(\mathscr{C})$ whenever $\mathscr{C}$ is a regular Mal'tsev category with coequalizers, relying on the result of [2] that Grpd($\mathscr{C}$) is a regular Mal'tsev category whenever $\mathscr{C}$ is so. An explicit description of the reflector $\mathrm{Grpd}^2(\mathscr{C}) \to$ 2-Grpd($\mathscr{C}$) is provided, which was not known in the varietal case, that reveals an interesting link with commutator theory of equivalence relations.

A category $\mathscr{C}$ with finite limits and finite colimits is called action representable if the functors $\mathrm{Act}(-, X)$ that map an object B to the set of its actions on X are representable, i.e., there exists an object $[X]$, called the actor of X, such that $\mathrm{Act}(B, X)$ and $\mathrm{Hom}(B, [X])$ are naturally isomorphic. Examples of action representable categories are given by Grp and Lie_R. The actor of a group G is given by its automorphism group $\mathrm{Aut}(G)$, while the actor of a Lie algebra L is given by its Lie algebra of derivations $\mathrm{Der}(L)$. If $\mathscr{C}$ is a semi-abelian category, the functor $\mathrm{Act}(-, X)$ is isomorphic to the functor $\mathrm{SplExt}(-, X)$ that maps an object B to the set of isomorphism classes of split extensions of B by X. It is shown in [3] that Grpd($\mathscr{C}$) is semi-abelian, action representable, algebraically coherent and with normalizers if and only if $\mathscr{C}$ is so. Using the fact that 2-Grpd($\mathscr{C}$) is coreflective and reflective in $\mathrm{Grpd}^2(\mathscr{C})$ if $\mathscr{C}$ is a semi-abelian category, we apply this result to show that 2-Grpd($\mathscr{C}$) shares the same properties if $\mathscr{C}$ does.

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Confluence of Term Rewriting Systems with Variable Binding

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Abstract. We consider the confluence of term rewriting systems for syntax with variable binding. Our contribution is a generalised confluence theorem in the spirit of Aczel [1] in the context of the model of second-order abstract syntax in the object-classifier topos [3, 2].

We recall from [3, 2] that a signature with variable-binding operators Σ induces a strong term monad T_Σ on $\mathbf{Set}^{\mathbb{F}}$, for $\mathbb{F}$ the category of finite cardinals and functions. We define a rewrite rule to be given by a pair of terms in $T_\Sigma(M)_0$ for a presheaf of meta-variables M. A rewrite rule ρ inductively defines a reduction relation $\leadsto_\rho \subseteq T_\Sigma(0) \times T_\Sigma(0)$ in $\mathbf{Set}^{\mathbb{F}}$. Closely following Aczel's approach, we present a coherence property for a rewrite rule ρ that is shown to yield the confluence of the reduction relation $\leadsto_\rho$. As an application, we examine the confluence of β-reduction in the λ-calculus.

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Premonoidal and Kleisli double categories

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Abstract.

Premonoidal categories appeared in Computer Science as a model that encodes the structure of effectful languages. Recently in [1] the notion of premonoidal bicategories was introduced covering semantic models that have more structure and supply more detailed and determined data. In this talk we give a double categorical version of the notion and the accompanying results.

We introduce two kinds of funny products on double categories and the corresponding closed representable multicategories, which we call *funny multicategories*, providing the category *Dbl* of double categories structures of closed funny monoidal products. We show that a monoid in one of the monoidal structures and a pseudomonoid in the monoidal 2-category $(Dbl_2, \square_2)$ induced by the other funny monoidal product are premonoidal double categories of certain kinds. We also describe binoidal structures induced by pseudodouble funny functors (multimaps for funny multicategories).

We prove that a premonoidal double category $\mathbb{D}$ is purely central if and only if its binoidal structure is given by a pseudodouble quasi-functor of two variables (a multimap for a Gray type of multicategory) if and only if $\mathbb{D}$ admits a monoidal structure. For such double categories we introduce pure center double categories and show that the monoidal structure on $\mathbb{D}$ extends to a one on the corresponding pure center. We differentiate a (general) center double category in which left and right centrality structures are not "sufficiently well" related and that henceforth does not present a monoidal structure.

Provided existence of certain companions, we prove a series of further results by applying companion-lifting of vertical structures into their horizontal counterparts. We firstly introduce a vertical strength t on a vertical double monad T on a monoidal double category, and also a horizontal strength s on a horizontal double monad S on a horizontally monoidal double category. We prove that vertical strengths induce horizontal strengths $\hat{t}$ on the induced horizontal double monads $\hat{T}$. We show that vertical strengths induce actions of the induced horizontally monoidal double category $(\mathbb{D}, \otimes, \hat{\alpha}, \hat{\lambda}, \hat{\rho})$ on the horizontal Kleisli double category $\mathbb{K}l(\hat{T})$. Morever, we prove that there is a 1-1 correspondence between horizontal strengths and extensions of the canonical action of $\mathbb{D}$ on itself (a canonical action consists of a horizontal action on the Kleisli double category and a horizontal icon). Finally, we show that for a bistrong vertical double monad T the Kleisli double category $\mathbb{K}l(\hat{T})$ is premonoidal. By the latter three results we lifted to double categories the corresponding bicategorical results from [1, 2]. We exhibit the advantage of working with vertical structures in double categories: companion-lifts of 2-cells appearing in the computations are globular 2-cells of certain type that satisfy any equation that can be comprised by their compositions. Consequently, for bicategories that are underlying bicategories of double categories with suitable companion-lifts much simpler proofs can be performed by applying companion-lifting of the vertical structures, as is exemplified in this work.

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Monilmorphisms and relative extensivity

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Abstract.
We introduce a new class of morphisms in a pointed category with kernels, which contains all monomorphisms and all null morphisms. We call a morphism $f\colon X \to Y$ in such a category a *monilmorphism* when for any two morphisms $g_1, g_2\colon W \to X$ we have:

$$[f \circ g_1 = f \circ g_2 \ \wedge\ \mathsf{ker}(f \circ g_1) = \mathsf{ker}(g_1) \ \wedge\ \mathsf{ker}(f \circ g_2) = \mathsf{ker}(g_2)] \implies [g_1 = g_2].$$

It is easy to see that in the category of sets and partial functions, monilmorphisms are precisely the injective partial functions. More generally, in any restriction category [1] with a restriction zero, the monilmorphisms are the same as restricted monics if and only if all restriction idempotents are monilmorphisms. On the other hand, in the category **Rel** of sets and relations as morphisms, a morphism is a monilmorphism if and only if it is a monomorphism or a zero morphism.

In the first part of the talk, we establish general properties of monilmorphisms and characterize them in various types of categories. In particular, we show that while every monilmorphism has the property that subobjects of its codomain can have at most one section, this property characterises monilmorphisms in any semi-abelian [4] category. In a number of particular semi-abelian categories (e.g. $\mathbf{Set}_*^{\mathrm{op}}$ and $\mathbf{Vect}_K$) monilmorphisms again reduce to just monos and zero morphisms. In any 0-regular variety, though, every morphism from an algebra with distributive subalgebra lattice is a monilmorphism. For example, in the category of groups, any homomorphism from a locally cyclic group is a monilmorphism.

In the second part of the talk, we discuss an application of monilmorphisms in the study of relative extensivity. In [3] it is shown that every monoidal sum structure $\oplus$ on a category can be described in terms of a certain kind of binary relation $\sqsubset$ on morphisms. Every morphism f is $\sqsubset$-preserving in the sense that $f_1 \sqsubset f_2 \Rightarrow ff_1 \sqsubset ff_2$. But, such a morphism might not be "totally honest" to its domain about all structure pertaining to $\oplus$ in its codomain. The morphisms that are totally honest in this sense are those where $ff_1 \sqsubset ff_2 \Rightarrow f_1 \sqsubset f_2$ — the $\sqsubset$-reflecting morphisms. Now, one can define extensivity relative to a morphism exactly as in [2] but with coproduct replaced by $\oplus$. One might require that all morphisms are extensive w.r.t. $\oplus$, but this is often too strong a condition on $\oplus$. Less demanding is that the extensivity requirement holds only for the $\sqsubset$-reflecting morphisms. This is the case, for example, in various categories of sets equipped with relational structure, where $\oplus$ is the stacking operation (every element of A is related to every element of B in $A \oplus B$). As we will show, monilmorphisms allow us to build examples of categories equipped with a sum structure $\oplus$ where $\sqsubset$-reflecting morphisms are exactly the morphisms that are extensive with respect to $\oplus$.

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In Regular Protomodular Categories with an Initial Object, Noetherian Objects are Hopfian and Closed Under Subobjects, Quotients and Extensions

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Abstract.

Classical Noetherian objects in algebra, such as modules over a ring, are characterized by the ascending chain condition on subobjects and exhibit key properties including being Hopfian and closed under submodules, quotients, and extensions. This work presents a categorical generalization of these concepts. Firstly, we extend the notion of Noetherian objects to the setting of regular protomodular categories (see [1]) with an initial object. Within this context, we demonstrate that the core properties are preserved: Noetherian objects, defined via stabilizing chains of subobjects, are Hopfian; any regular epimorphism $x \to x$ is an isomorphism for a Noetherian object x. Here assumptions can be relaxed to local ones. We further demonstrate that the class of Noetherian objects is closed under subobjects, regular quotients, extensions, and in the pointed case under finite products.

Secondly, we show that these results established for regular protomodular categories can be further generalized by abstracting the essential mechanisms into the language of stable weak factorization systems (E, M). We define (E, M)-Noetherian objects based on the stabilization of sequences of M-subobjects. The central finding is that the Hopfian property, all morphisms $x \to x \in E$ are isomorphisms (with assumptions on the initial object), and the closure properties hold even in this broader setting, provided the category has pullbacks and the system (E, M) possesses suitable characteristics, such as stability, weak images (weakening of the lifting condition) and a variation of the short-five-lemma, which mimic the behavior characterizing protomodular setting. This factorization system perspective isolates the key structural ingredients required for Noetherian-like behavior. The framework unifies classical results and applies widely, particularly within regular protomodular categories including loops, groups, rings, R-modules, Lie-algebras, crossed modules, Heyting algebras and hoops.

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Enrichment and families over virtual double categories

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Abstract.

Enriched category theory gives rise to a 2-functor Enr from a suitable 2-category of enrichment bases to the 2-category 2-$\mathbf{CAT}$ of 2-categories, sending each base $\mathcal{V}$ to the 2-category $\mathrm{Enr}(\mathcal{V}) = \mathcal{V}$-$\mathbf{Cat}$ of all (small) $\mathcal{V}$-categories. Classically, monoidal categories are taken as the enrichment bases, but there are several extensions taking e.g. bicategories, pseudo double categories, multicategories, and virtual double categories [2] as enrichment bases. In this talk, we will show that the 2-functor Enr becomes a *parametric right 2-adjoint* if we take the virtual double categories as the enrichment bases. That is, letting $\mathbf{VDBL}$ be the 2-category of virtual double categories, we show that the 2-functor $\mathrm{Enr}_1 \colon \mathbf{VDBL} \to 2\text{-}\mathbf{CAT}/\mathrm{Enr}(1)$, induced from the 2-functor $\mathrm{Enr}\colon \mathbf{VDBL} \to 2\text{-}\mathbf{CAT}$ and the terminal object 1 in $\mathbf{VDBL}$, is a right 2-adjoint.

In more detail, we first note that the 2-functor $\mathrm{Enr}\colon \mathbf{VDBL} \to 2\text{-}\mathbf{CAT}$ can be decomposed as

$$\mathbf{VDBL} \xrightarrow{\mathbb{M}\mathbf{at}} \mathbf{VDBL} \xrightarrow{\mathbb{M}\mathbf{od}} \mathbf{VDBL}_{\mathrm{n}} \xrightarrow{\mathsf{V}} 2\text{-}\mathbf{CAT}, \text{ where}$$

- $\mathbf{VDBL}_{\mathrm{n}}$ is the 2-category of virtual double categories with chosen horizontal units and virtual double functors preserving the chosen horizontal units on the nose,
- $\mathbb{M}\mathbf{at}$ maps each $\mathbb{D} \in \mathbf{VDBL}$ to the virtual double category $\mathbb{M}\mathbf{at}(\mathbb{D})$ of *matrices* in $\mathbb{D}$,
- $\mathbb{M}\mathbf{od}$ maps each $\mathbb{D} \in \mathbf{VDBL}$ to the virtual double category $\mathbb{M}\mathbf{od}(\mathbb{D})$ of *horizontal monads* in $\mathbb{D}$ (which is naturally equipped with chosen horizontal units) [3, 1], and
- V maps each $\mathbb{D} \in \mathbf{VDBL}_{\mathrm{n}}$ to its *vertical 2-category* $\mathsf{V}(\mathbb{D})$ [1].

We then observe that, on the one hand, both $\mathbb{M}\mathbf{od}$ and V are right 2-adjoints, and on the other hand, $\mathbb{M}\mathbf{at}$ is the *polynomial 2-functor*

$$\mathbf{VDBL} \xrightarrow{T^*} \mathbf{VDBL}/(\mathbf{Set}_*)_{\mathrm{hc}} \xrightarrow{\Pi_P} \mathbf{VDBL}/\mathbf{Set}_{\mathrm{hc}} \xrightarrow{\Sigma_S} \mathbf{VDBL}$$

induced by a suitable polynomial

$$1 \xleftarrow{T} (\mathbf{Set}_*)_{\mathrm{hc}} \xrightarrow{P} \mathbf{Set}_{\mathrm{hc}} \xrightarrow{S} 1$$

in $\mathbf{VDBL}$, and hence is a parametric right 2-adjoint. A closely related polynomial in $\mathbf{VDBL}$ induces the 2-functor $\mathbb{F}\mathbf{am}$ for the family construction for virtual double categories.

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Categorical-algebraic characterisations of Lie algebras

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Abstract.

Let $\mathfrak{M}$ be a non-abelian variety of non-associative algebras over a field of characteristic zero. In [2] we proved that $\mathfrak{M}$ is *locally algebraically cartesian closed* if, and only if it is the variety of Lie algebras. In this way, we provided a characterisation of Lie algebras amongst all varieties of non-associative algebras in a purely categorical way, using *algebraic exponents*.

The key ideas behind this result involve studying the preservation of coproducts for every object B, of the endofunctor $B\flat(-)\mathfrak{M} \to \mathfrak{M}$, that sends an object X to the kernel of the canonical split extension

$$B\flat X \longrightarrow B + X \underset{i_B}{\overset{\langle 1,0\rangle}{\rightleftarrows}} B$$

Inspired by this result we found another characterisation of Lie algebras in [1]:

Theorem 1. *Suppose that $\mathfrak{M}$ is a non-trivial variety of non-associative algebras over a field of characteristic zero satisfying the following two conditions:*

- *it is Nielsen-Schreier;*
- *for every normal object I, I^2 is also normal.*

Then $\mathfrak{M}$ is the variety of Lie algebras.

In this talk we will begin with a conceptual overview of this result, together with its motivation and its origins. We will relate it with the previous characterisation that used algebraic exponents, linking the injectivity and surjectivity of the comparison map

$$B\flat X + B\flat Y \longrightarrow B\flat(X + Y)$$

with the properties of the theorem. Finally, we will discuss the proof techniques, which apart from the categorical-algebraic methods, they also involve homological and computational algebra, together with Gröbner bases for operads.

Joint work with Vladimir Dotsenko (Université de Strasbourg)

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Pretorsion Theories on $(\infty, 1)-$Categories

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April 2025

1 Abstract

In the 1960s, Spencer Dickson, in [3], axiomatizing properties of the category of abelian groups, presented a notion of *torsion theory* on abelian categories, which was soon generalized beyond the abelian setting (see, among many others, [1], [2]), and even, recently, in [4] and particularly [5], to general (and not necessarily pointed) $1-$categories. Classically, a *pretorsion theory* on a category $\mathbf{C}$ is a pair of full replete subcategories $(\mathbf{T}, \mathbf{F})$ such that every morphism between them factors through their intersection, $\mathbf{Z} := \mathbf{T} \cap \mathbf{F}$, and that there is a notion of short exact sequence consisting of a $\mathbf{Z}-$kernel and a $\mathbf{Z}-$cokernel that one may associate to every object in $\mathbf{C}$. A *torsion theory* from this perspective is a pretorsion theory where $\mathbf{Z} = \varnothing$. Pretorsion theories satisfy multitudinous properties, including that $\mathbf{T}$, $\mathbf{F}$ and $\mathbf{Z}$ are closed under certain extensions, that $\mathbf{Z}-$kernels and -co-kernels are respectively monomorphisms and epimorphisms, and that $\mathbf{T}$ and $\mathbf{F}$ are respectively coreflective and reflective subcategories of $\mathbf{C}$.

Here we will propose a notion of pretorsion theory for $(\infty, 1)-$categories, compatible with the aforementioned classical one under the taking of the homotopy category $h\mathscr{C}$ of the $(\infty, 1)-$category $\mathscr{C}$ upon which the pretorsion theory is situated. We shall then show that $(\infty, 1)-$categorical pretorsion theories satisfy some, but not all, of the properties fulfilled by their classical counterparts.

There are a notion of normal torsion theory for stable $(\infty, 1)-$categories [6], as well as a version of two-dimensional torsion theory [8] available in the literature, and we will discuss the the compatibility and relationship of our construction with these two by looking in particular at $(\infty, 1)-$torsion theories in our framework as well as its behavior under truncation.

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Actions of partial groups, and the higher Segal conditions

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Abstract.
Partial groups were introduced in [1] and played a key role in Chermak's proof of the existence and uniqueness of centric linking systems for saturated fusion systems, a major recent result in p-local finite group theory. A partial group consists of a (suitably compatible) collection of partially-defined n-ary multiplications on a set; this structure can concisely be described as a symmetric simplicial set for which the Segal maps are monomorphisms and which has a single vertex [5]. The most important class of partial groups are the *localities*, each of which comes equipped with an 'action' by conjugation on a certain subcollection of its set of p-subgroups.

Heretofore, the notion of action of partial groups has proved elusive. We'll propose a simple general definition in the symmetric sets framework, inspired by the fibrational perspective in category theory. This will be suitable for capturing a number of expected examples, including the action of a locality on its special collection of subgroups, and more generally for partial groups arising from a partial action of a group [3]. These are examples of *characteristic actions*, where the action itself controls which multiplications are valid in the partial group.

Our main application of characteristic actions is to a connection between partial groups and the d-Segal spaces of Dyckerhoff–Kapranov [2]. The higher Segal conditions are a series of exactness conditions for simplicial spaces coming from certain triangulations of cyclic polytopes. These generalize Segal spaces, and have applications (for $d = 2$, when they are also called *decomposition spaces* [4]) in representation theory, K-theory, geometry, combinatorics, and elsewhere, and are closely connected to ∞-operads and to categories with multivalued composition. For a given partial group, one naturally might wonder if it is d-Segal for some d. The *degree* of a partial group is defined to be the least integer k for which it is $(2k-1)$-Segal, so that the degree 1 partial groups are precisely the groups. We'll discuss why characteristic actions are a key ingredient in degree computations.

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Natural equivalences and weak invertibility of higher-categorical contexts

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Abstract.
A *context* in a higher category is a "pasting diagram with a hole"—an arrangement of cells around a boundary, such that inserting any compatible cell into the boundary produces a valid pasting diagram. A result expected to hold across models of higher categories is that, if a context E is built out of internal *equivalences* in a suitable way, then a well-formed equation of the form $\mathsf{E}x = b$ should have a unique solution up to equivalence. A proof of this fact exists for strict ω-categories [1], but parts of this proof do not even typecheck in a model with non-strict units.

We study this problem [2] in the setting of diagrammatic sets, which capture the "raw" combinatorics of pasting diagrams. The notion of equivalence in terms of *coinductive weak invertibility* can be formulated internally to any diagrammatic set. Diagrammatic sets support models of (∞, n)-categories, as well as faithful nerve functors from a variety of higher-categorical structures, so results proved for diagrammatic sets have wide model-independent applicability.

We introduce a coinductive relation $\mathsf{C} \simeq \mathsf{D}$ of *natural equivalence* between contexts in a diagrammatic set, and prove that it is a congruence with respect both to composition of contexts and to pasting in codimension 1. Moreover, we prove that certain natural equivalences involving units and unitors exist in every diagrammatic set. These facts give us access to a powerful algebraic calculus of natural equivalences of contexts. Finally, we study the class of contexts built out of weakly invertible diagrams, which we call *weakly invertible contexts*, and prove our main theorem: every weakly invertible context E admits a two-sided weak inverse, that is, a weakly invertible context E^* such that $\mathsf{E}\mathsf{E}^* \simeq -$ and $\mathsf{E}^*\mathsf{E} \simeq -$, where $-$ denotes the trivial context. This implies the result that we were aiming for: every well-formed equation $\mathsf{E}x = b$ admits the weakly unique solution $x \coloneqq \mathsf{E}^*b$.

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Generalised ultracategories and conceptual completeness of geometric logic (work in progress)

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Abstract.

Conceptual completeness is an important result in first order categorical logic, it states that, with an additional structure on the category of models of coherent first-order theory, describing categorically the notion of ultraproduct, it is possible to establish a syntax semantics equivalence between a certain class of functors called ultrafunctors from the category of models of a coherent theory to Set, and a pretopos (which can be thought of as a completion of the syntactic category of this theory). Towards showing this result Makkai introduce the notion of ultracategories [1] and ultrafunctors, these are categories equipped with an ultraproduct functor (together with data and coherence). More recently Lurie [2] reintroduced ultracategories and showed a version of conceptual completeness stating that there is an equivalence between a certain class of functors called left ultrafunctors from the category of points of a topos (models of the coherent theory classified by this topos) to Set and the Topos itself.

We want to extend Makkai and Lurie's results to any topos with enough points, the first obstruction is that the category of points of such toposes do not have a canonical notion of ultraproduct. Toward this, we introduce a new notion of generalised ultracategories, where the ultraproduct may not exist, but we may find instead the "representable" at the ultraproduct. It turns out that generalised ultracategories are connected to topological spaces the same way usual ultracategories are connected to compact Hausdorff spaces. We can assign to every topological space a unique generalised ultrastructure coming from the notion of ultrafilter convergence. We can expand on this, and show that the generalised ultrastructure of any generalised ultractegory can be fully determined by "covering" such category with suitable categories of points of topological spaces. There is a similar result for Toposes with enough points, stating that they can be expressed as the 2-colimit of certain "logical" topological groupoids [3]. We relate these facts together to deduce a conceptual completeness theorem stating that for any two toposes with enough points E and $E^{'}$ there is an equivalence between $\mathrm{Geom}(E, E^{'})$ and $\mathrm{Left-ultrafunctors}(\mathrm{Points}_E, \mathrm{Points}_{E'})$. Here Left ultrafunctors are the generalised version of left ultrafunctors that reduces to the ordinary ones in the case of coherent toposes. This result reduces to a one similar to Lurie's one if we replace the topos $E^{'}$ by the classifying topos of the theory of objects.

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Unfolding of symmetric monoidal (∞, n)-categories

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Abstract.
In Lurie's article on the Cobordism Hypothesis [1], he discusses a description of symmetric monoidal (∞, n)-categories with duals for objects and certain adjoints as "chain complexes" of symmetric monoidal ∞-categories with duals, but does not give a proof. This gives a surprisingly simple description of symmetric monoidal (∞, n)-categories with duals that should be useful both in connection with extended TQFTs and in stable homotopy theory. We will give a proof of this "unfolding" equivalence in forthcoming work, based on a general description of closed symmetric monoidal $\mathcal{V}$-enriched ∞-categories as lax symmetric monoidal functors to $\mathcal{V}$. In this talk I will explain how this comparison works, focusing on the simplest cases (for $n = 2$ and 3).

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Notions of Cauchy (co)completeness for normed categories

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Abstract.

Lawvere's famous 1973 paper [3] has inspired numerous works connecting metric theory and category theory. One of the lesser known suggestions hinted at in [3] is the notion of *normed category*: a category X where each hom-set $\mathsf{X}(A,B)$ comes equipped with a *norm* $\mathsf{X}(A,B) \to [0,\infty]$, subject to suitable axioms. Normed categories can be viewed as categories enriched in the closed symmetric monoidal category of *normed sets*, briefly mentioned in [3] and explicitly described in [1].

In continuation of Walter Tholen's talk at CT2024 (see also [2]), we also consider norms which take values in a quantale $\mathcal{V}$ (and not only in $[0,\infty]$) and recall basic properties of the category of $\mathcal{V}$-normed sets and of $\mathcal{V}$-normed categories, respectively, as well as the notions of normed colimit and Cauchy cocompleteness for $\mathcal{V}$-normed categories. We illustrate these notions with various examples of (large) normed categories. Our main focus is on the calculus of normed distributors. Besides the description of normed colimits as weighted colimits for a certain class of distributors, we investigate the classic notion of "Cauchy completeness *à la* Lawvere", that is: we call a normed category *Lawvere complete* whenever every left-adjoint normed distributor into it is representable. In particular, we provide a characterisation of Lawvere complete normed categories via an idempotent completeness condition and an approximation condition. Finally, we relate these notions to the corresponding concepts for $\mathcal{V}$-enriched categories and for ordinary categories.

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Topoi of Automata

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Abstract.

In this talk, based on [2] and ongoing work, we explore a topos-theoretic perspective on automata theory and regular languages.

We begin with the naive presheaf topos Σ-Set, the category of presheaves on the free monoid generated by a fixed alphabet Σ. The stalk functor at the canonical point of Σ-Set reveals its connection to the coalgebraic and functorial [1] approaches to automata theory.

Next, we construct a Grothendieck topos $\Sigma\text{-Set}_{\text{o.f.}}$ that fully encodes the structure of regular languages. We then explain how this topos arises naturally from four distinct perspectives on regular language theory: deterministic finite automata (DFA), finite monoids, the Myhill–Nerode theorem, and profinite topology.

If time allows, we show how hyperconnected geometric morphisms from $\Sigma\text{-Set}_{\text{o.f.}}$ describe the syntactic monoid of a language, and argue that many "geometric invariants" remain to be uncovered in this context.

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Internal categories, algebraic model structures and type theory

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Abstract.

Internal category theory has witnessed an increase in interest recently, partially due to its connection with double category theory, 2-dimensional universal algebra and alternative foundations of mathematics. A natural line of enquiry is to investigate to what extent results from (small) category theory carry over to internal category theory. In this vein, Everaert, Kieboom and Van der Linden have shown that the canonical model structure on **Cat** (where weak equivalences are equivalences of categories and fibrations are isofibrations) can be generalised to $\mathbf{Cat}(\mathcal{E})$, where $\mathcal{E}$ satisfies suitable assumptions [2].

In this talk I will explain some recent work [4] in which the aforementioned model structure on $\mathbf{Cat}(\mathcal{E})$ is upgraded to an algebraic model structure [6]. To prove that such an algebraic model structure exists on **Cat** is not too difficult, but to unwind the explicit characterisation of this is more subtle and is both easier to do and more general if we work in $\mathbf{Cat}(\mathcal{E})$. This involves giving a characterisation of all the classes of maps involved— for example, the algebraic fibrations are given by cloven isofibrations. This falls out from the construction of the (co)monads involved which equip algebraic structure to any map freely, and translations between different types of algebraic structure.

Given a finitely complete and cocomplete 2-category $\mathcal{K}$, there is a model structure on it by defining the weak equivalences and fibrations representably [5]. Thanks to work by Bourke [1], it is possible to leverage our result and give a list of elementary axioms on a 2-category $\mathcal{K}$ such that we can upgrade this model structure to an algebraic model structure.

As an application, I will explain how we can use this to construct an algebraic internal groupoid model of Martin-Löf type theory. This recovers Hofmann and Streicher's groupoid model [3] when working with groupoids internal to **Set** and forgetting the algebraic structure. To conclude, I will apply these results to different examples of $\mathcal{E}$ such as categories of (pre)sheaves, the effective topos and categories which themselves model Martin-Löf type theory. In some cases, this gives an improved perspective on results that have been previously considered in the literature; in other cases, it gives novel results.

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A double barreled approach to composing dynamical systems and their morphisms

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Abstract.

Motivated by developing a representability theory for open dynamical systems, we've investigated a *double barreled* approach to *loose* bimodules between double categories.

Joyal's "barrels" present bimodules $M : A^{\mathsf{op}} \times B \to \mathsf{Set}$ between categories as functors $p : C \to \Delta[1]$ into the walking arrow for which $p^{-1}(0) = A$ and $p^{-1}(1) = B$ and $p^{-1}(0 \to 1) = \coprod_{a,b} P(a,b)$; C is the *collage* of the bimodule. When categorifying to double categories, we have two choices for generalization; we could look at barrels over the loosely discrete double category $t\Delta[1]$ consisting of one *tight* arrow, which would correspond to Paré-style bimodules $\mathbb{A}^{\mathsf{op}} \times \mathbb{B} \to \mathbb{S}\mathsf{pan}(\mathsf{Set})$, or we could look at barrels over the tightly discrete double category $\ell\Delta[1]$ which would correspond to *loose* bimodules — bimodules whose "heteromorphisms" go in the loose (a.k.a. pro) direction. In this talk we'll investigate the latter.

We may easily define a 2-category of loose bimodules as the slice $\mathcal{D}\mathsf{bl} \downarrow \ell\Delta[1]$, where $\mathcal{D}\mathsf{bl}$ is the 2-category of double categories, pseudo-functors, and tight transformations. We then investigate the functoriality of *restriction* of loose bimodules by constructing a product-preserving pseudo-functor

$$\mathsf{Res} : \mathcal{N}\mathsf{iche} \to \mathcal{D}\mathsf{bl} \downarrow \ell\Delta[1]$$

where $\mathcal{N}\mathsf{iche}$ is a 2-category of "niches" defined as the pullback:

$$\begin{tikzcd}
\mathcal{N}\mathsf{iche} \arrow[r] \arrow[d] \arrow[dr, phantom, ">"] & \mathcal{D}\mathsf{bl} \downarrow \ell\Delta[1] \arrow[d, "{(d_1^*, d_0^*)}"] \\
2\mathcal{C}\mathsf{at}^{\mathsf{colax,conj}}(\Delta[1], \mathcal{D}\mathsf{bl}) \times 2\mathcal{C}\mathsf{at}^{\mathsf{colax,comp}}(\Delta[1], \mathcal{D}\mathsf{bl}) \arrow[r, "d_0^* \times d_0^*"'] & \mathcal{D}\mathsf{bl} \times \mathcal{D}\mathsf{bl}
\end{tikzcd}$$

where the 2-categories in the bottom left are colax commuting squares in $\mathcal{D}\mathsf{bl}$ whose colaxitors are conjoint (resp. companion) *commuter* transformations in the sense of Paré. As a corollary, we conclude (using Arkor-Bourke-Ko's wonderful *symmetry of internalization*) that the restriction of a symmetric monoidal loose bimodule M by lax symmetric monoidal double functors F_0 and F_1 yields a symmetric monoidal loose bimodule $M(F_0, F_1)$ so long as the laxitors of F_0 are conjoint commuter cells and those of F_1 are companion commuter cells.

We'll end by using the above to construct a number of symmetric monoidal loose right modules of *open dynamical systems* acted upon by symmetric monoidal categories of *composition processes*, generalizing constructions of Schultz, Spivak, and Vasilakopoulou to include morphisms of systems with different interface and laying the ground for a representable account of system behavior. These constructions can be performed *pseudo-functorially* using the general span construction for Haugseng-Hebestreit-Linskens-Nuiten *adequate triples*. To account for monadic non-determinism, we show that an suitable 2-category of adequate triples has Kleisli objects. The resulting span construction generalizes Leinster's construction of T-spans to suitable monads on adequate triples.

Bicolimit Presentations of Type Theories

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Abstract. Presentations of algebras are a way of defining algebras using generators and relations that these generators satisfy. A more categorical point of view is that every algebra can be expressed as a coequaliser of free algebras. In this talk, we explore how similar ideas can be used in semantics of type theory.

In his PhD thesis [1], Taichi Uemura proposed a definition of type theories that allows functorial semantics of type theory: a type theory is a small category with finite limits and a class of arrows satisfying some properties. Models are then functors preserving all the structure.

We examine the 2-categorical aspects of this approach. Using tools developed in [2] and [3], we

- define a notion of signature that allows us to freely generate type theories by some simpler data (this is formalised as an existence of a free-forgetful biadjunction);
- explain that type theories can be glued together via bicolimits.

Luckily, (a big class of) bicolimits and the 'free generation' interact well with our notion of semantics: Whenever we have a bicolimit of freely generated type theories (i.e. if we have a bicolimit presentation of a type theory), we can understand its semantics as long as we understand the semantics of the generating data.

In this way, we can construct type theories whose semantics recovers various flavours of semantics of dependent type theory. For example, we can create a type theory whose models are natural models [4] or a type theory whose models are natural models with Π-types.

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Representables in fuzzy category theory

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Abstract.

Fuzzy category theory describes category-like structures in which potential objects and potential morphisms are respectively objects and morphisms only to a certain degree [1]. By easing the requirements on objects and morphisms, we can create models for situations that cannot be directly described using the tools of classical or enriched category theory. This approach enables the study of structures with varying degrees of existence, offering a localized perspective on objects and morphisms while maintaining an otherwise context-free setting.

Enriching categories over the symmetric monoidal closed category of L-sets [2], we effectively assign each morphism in a category a membership degree from a given lattice L. Since L-set enriched categories naturally embed into fuzzy categories—where both objects and morphisms can be assigned membership degrees—this motivates the exploration of representations arising from fuzzy functors. By examining fuzzy functors mapping into the fuzzy category $\boldsymbol{FL}$-**Set**, we extend the classical theory of representable functors into the realm of fuzzy categories.

In this talk, we establish conditions under which such fuzzy functors can be considered representable, particularly in the context of threshold categories, which are obtained by fixing an idempotent element in the underlying quantale structure. We analyze the relationship between fuzzy functor representations and threshold categories, demonstrating that representable fuzzy functors naturally induce representable enriched $\boldsymbol{L}$-**Set**-functors when viewed within an enriched categorical framework.

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Positively closed topos-valued models

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Abstract.
This talk is about models of coherent theories (T) internal to Grothendieck toposes ($\mathcal{E}$).

One possible motivation is that certain classical notions of mathematics can be captured by internal models. For example, if T is the theory of local rings and $\mathcal{E} = Sh(X)$ for some space X, then T-models in $\mathcal{E}$ are locally ringed spaces with underlying space X. So if we can lift positive model theory to the topos-valued setting, then we gain new tools to study such objects.

A model of T internal to $\mathcal{E}$ is the same as a coherent functor $\mathcal{C}_T \to \mathcal{E}$, where $\mathcal{C}_T$ is the syntactic category. As a result, topos-valued positive model theory can be seen as the study of coherent functors $\mathcal{C} \to \mathcal{E}$ where $\mathcal{C}$ is some coherent category.

A key concept in positive model theory is the notion of a positively closed model. A model M is positively closed if every outgoing homomorphism reflects the validity of positive existential formulas. In the functorial language: if every outgoing natural transformation from $M : \mathcal{C} \to \mathcal{E}$ to some other coherent functor $N : \mathcal{C} \to \mathcal{E}$ is mono-cartesian. This is a global notion, as it concerns many other models.

A coherent functor $M : \mathcal{C} \to \mathcal{E}$ is strongly positively closed if for any monomorphism $u \hookrightarrow x$ in $\mathcal{C}$ we have $Mx = Mu \cup \bigcup_{v \hookrightarrow x:\ v \cap u = \emptyset} Mv$. This is a local notion as it depends only on M. It is well-known that for $\mathcal{E} = \mathbf{Set}$ the positively closed and the strongly positively closed models coincide.

Lurie's [3, Lecture 16X, Theorem 11] says that if $\mathcal{C}$ is coherent with finite disjoint coproducts, then any $\mathcal{C} \to \mathbf{Set}$ regular functor can be factored uniquely as $\mathcal{C} \xrightarrow{M} Sh(B, \tau_{coh}) \xrightarrow{\Gamma} \mathbf{Set}$ where B is a Boolean algebra, τ_{coh} is the topology formed by finite unions, M is coherent and Γ is global sections (see also [2, Theorem 4.6]). In particular a $Sh(B, \tau_{coh})$-valued model can be uniquely recovered from its global sections. This makes $\mathcal{E} = Sh(B, \tau_{coh})$ a good test case.

I will give examples of positively closed but not strongly positively closed $Sh(B, \tau_{coh})$-valued models where B is a complete Boolean algebra.

I will prove that if B is complete, $Sh(B, \tau_{coh})$-valued models satisfy some form of compactness: they can realize types. This is a major ingredient for proving that although for such models positively closed is not equivalent to strongly positively closed, it is equivalent to an alternative local property.

The talk is based on [1].

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Compact, Hausdorff and locally compact locales in toposes.

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Abstract.
Frames that are internal in a presheaf topos are presheaves of frames satisfying further conditions [2], C 1.6.9. For example, the internal completeness of the frame L is captured by the existence, for each $f\colon b \to a \in \mathcal{C}$, of a left adjoint $\Sigma_f \dashv Lf\colon La \to Lb$ satisfying the Frobenious law. As a result, there is a forgetful functor $U\colon \mathrm{Frm}[\mathcal{C}^{op}, \mathbf{Set}] \longrightarrow [\mathcal{C}^{op}, \mathbf{Frm}]$. It can be derived from [3] that this functor has a left adjoint $\mathfrak{z} \dashv U$ with sections $\mathfrak{z}L(a) = \{(u_f) \in \prod_{\vartheta_1 f = a} L(\vartheta_0 f) \mid \forall g\colon b \to a\ (L(g)(u_f) \leq u_{fg})\}$. This same description arises in [1] as a way to universally turn a lax Pos-enriched natural transformation into a strict one and allows us to determine the sections of internal posets of subobjects and of ideals in presheaf toposes, while it gives one leg of the equivalence between the categories of presheaves of compact Hausdorff locales and internal such locales in a presheaf topos. As this equivalence is not obtained object-wise, we investigate related conditions in terms of properties of the sections of the internal frame.

Compactness of the frame L amounts to the equalizer of the supremum, and the constantly equal to the top element, maps $idlL \to L$, being $\{L\}$. From that it can easily be derived that the sections of a compact internal frame are compact. Local compactness amounts to the existence of a map $\Lambda\colon L \to idlL$, left adjoint to the supremum map, and from that we can derive a section-wise left adjoint to the supremum map, $\lambda_a\colon La \to idl(La)$ yielding the local compactness of the sections of L. It can further be calculated that the transition maps preserve the way-below relation of the sections, equivalently the localic transition maps are finitary. Concerning the Hausdorff condition, i.e the closedness of the localic diagonal $L \to L \times L$, the preservation of coproducs by $\mathfrak{z}$ and properties of closed maps give that if each section La is Hausdorff then L is internally Hausdorff, while if $\mathfrak{z}L$ is Hausdorff so is each section La. We discuss further the sufficiency of the conditions on the sections for the internal frame to be compact or locally compact and relevant conditions for sheaf toposes. For example, for a proper inclusion of a sheaf subtopos, compactness in the smaller topos is equivalent to that in the larger topos, so one still obtains compactness section-wise.

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Codiscrete cofibrations vs. iterated discrete fibrations for (∞, ℓ)-profunctors and ℓ-congruences

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Abstract.
The exactness properties of 2-categories [2] are formulated with simplicial kernels and codescent objects of cateads (or 2-congruences), internal categories whose underlying graph is a discrete two-sided fibration. For higher categories, this notion branches naturally into two different generalisations. On the one hand, the $(\ell-1)$-categorical fibration classifiers that should characterise $(\infty, \ell+1)$-topoi suggest looking at $(\ell+1)$-congruences (in an $(\infty, \ell+1)$-category $\mathfrak{K}$, for $\ell \in \mathbb{N} \cup \{\omega\}$) as internal categories whose underlying graphs are $(\infty, \ell-1)$-categorical two-sided fibrations, which requires working with a lax (or Gray-enriched) version of the kernel/codescent objects [3].

On the other hand, the use of lax limits and colimits can be avoided by defining ℓ-cateads as internal ℓ-categories whose underlying ℓ-graphs are iterated discrete two-sided fibrations; for this I will introduce cellular kernels and codescent objects of internal higher category objects, inspired by the formal methods of [1], making full use of the enrichment over $(\ell-1)$-categories to encode the higher structures in (strong) weighted limits. The object of this talk will then be to report on work in progress on the comparison between these two types of fibrations.

Since the $(\infty, \ell-1)$-categorical two-sided fibrations in $\mathfrak{K} = (\infty, \ell)\text{-}\mathfrak{Cat}$ precisely encode (∞, ℓ)-profunctors, the ideas of [4] on enriched profunctors indicate that they should correspond to codiscrete two-sided-cofibrations in $\mathfrak{K}$. In fact these will serve as the middle point between the two types of fibrations: writing $\mathfrak{Corr}_d(\mathfrak{K})$ for the ∞-category of d-iterated spans in $\mathfrak{K}$, we have a pair of adjunctions

$$\mathfrak{Corr}_1(\mathfrak{K}) \rightleftarrows \mathfrak{Corr}_1(\mathfrak{K}^{\mathrm{op}}) \leftrightarrows \mathfrak{Corr}_\ell(\mathfrak{K})$$

where the left adjunction is given by lax cocomma and comma, while the right one is given by a "bipartite" version of the cellular codescent objects and kernels, using higher commas. I will explain why, at least in $\mathfrak{K} = (\infty, \ell)\text{-}\mathfrak{Cat}$, the latter adjunction restricts to an equivalence between codiscrete cofibrations and iterated discrete fibrations.

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Categories for industrial planning

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Abstract. Category theory provides a way of thinking in solving many complex problems. We intend to explain an application on industrial networks, flows, and their data organization which was a part of two long-term applied projects solved in recent years.

The networks consist of processes, stacks, and channels, and those are aggregated into breakdown structures. Such organization naturally define a structure of dagger compact category and production schemes are described as certain diagrams. Namely, the tensor product aggregates resources and the dagger enables to define backward flows.

The model is mostly inspired by that of quantum protocols introduced by Abramsky and Coecke [1] and provides an interesting link between the two areas. For example, we will present a production analog of the double-split experiment. However, "logic" of the production networks is arise from classical finite relations (and not finite-dimensional Hilbert spaces). This provides a direct interpretation of categorical operations as clauses in relational databases.

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Linearly Distributive Fox Theorem

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Abstract.

Linearly distributive categories (LDC), introduced by Cockett and Seely to model multiplicative linear logic, are categories equipped with two monoidal structures that interact via linear distributivities [2]. A seminal result in monoidal category theory is the Fox theorem, which characterizes cartesian categories as symmetric monoidal categories where each object is equipped with a comonoid structure [3]. The aim of the current work is to extend the Fox theorem to LDCs and characterize the subclass of LDCs, whose tensor structure is cartesian and parr structure is cocartesian, known as cartesian LDCs.

As we will discuss in this talk, we must restrict our attention to LDCs which additionally satisfy the medial logical rule, named medial linearly distributive categories. The medial rule has appeared frequently in various deep inference systems [5]. It has equally been crucial in certain developments of categorical semantics for classical logic [4, 6]. Within monoidal category theory, the medial rule is better known as an instance of the interchange law of duoidal categories [1]. Indeed, medial LDCs can be thought of as the appropriate structure at the intersection of LDCs and duoidal categories.

Mirroring the cocommutative comonoids of the Fox theorem, a medial LDC induces a cartesian LDC by constructing the category of biccommutative medial bimonoids. The concept of medial linear functors, and medial linear transformations are equally introduced and an adjunction between the 2-categories of medial and cartesian LDCs is proved, named the *linear distributive Fox theorem.*

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The formal theory of vector fields for tangentads

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Tangent category theory, as introduced by Rosický [5] and later revisited by Cockett and Cruttwell [1], is a well-established categorical framework for differential geometry. In [4], a formal approach was adopted to provide a genuine Grothendieck construction in the context of tangent category theory, by introducing **tangentads**, previously named tangent objects. A tangentad is to a tangent category as a formal monad [6] is to a monad on a category. Since tangent category theory captures important geometric notions such as vector fields, it is natural to wonder whether or not these constructions can be lifted to the formal context. In particular, one wonders how to formalize important concepts such as vector fields for tangentads.

In this talk, I discuss the formal notion of tangentads. We present numerous examples of tangentads, such as (split) restriction tangent categories [1], tangent fibrations [2], and tangent monads [3]. I also introduce a formal construction for vector fields on tangentads by highlighting their universal property for tangent categories. I show how to recover the usual operations between vector fields, such as the Lie bracket. Moreover, I prove the existence of vector fields in the presence of PIE limits.

The paper can be found at `https://arxiv.org/abs/2503.18354`.

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The magnitude of a presheaf

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Abstract.
There is now a substantial literature on magnitude and magnitude homology [2]. The magnitude of an enriched category is the canonical invariant of its size. With different bases of enrichment, it recovers such classical invariants as cardinality, Euler characteristic, dimension, volume and surface area. Magnitude homology is a categorification of magnitude, in the sense that it is a homology theory of enriched categories whose Euler characteristic is magnitude (under hypotheses). It has been studied especially intensively for metric spaces and graphs.

But all of this existing work is on the magnitude and magnitude homology of enriched *categories*. Here, I will introduce the magnitude of enriched *presheaves*. I will explain the size-like properties of this invariant and how several existing measures of size naturally arise as instances of the general concept. In particular:

- The basic result on the magnitude of presheaves unifies elementary counting formulas for the cardinality of a colimit of sets.
- Entropy, relative entropy and conditional entropy for finite probability distributions all arise as special cases of the notion of the magnitude of a presheaf.
- Prime counting functions, in both number-theoretic and ring-theoretic contexts, also arise naturally as special cases.
- The concept of the magnitude of a presheaf allows one to give a categorical account of the PDE methods that have been successfully applied to calculate the magnitude of certain subsets of Euclidean space (as in [1], for instance).

Finally, I will introduce the dual concept, the comagnitude of a presheaf, which has an attractive interpretation in terms of the expected cardinality of the limit of a random presheaf.

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Extensional concepts in intensional type theory, revisited

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Abstract.

In his Ph.D. dissertation, Hofmann [Hof95] constructs an interpretation of extensional type theory in intensional type theory, subsequently proving a *conservativity* result of the former over the latter extended by functional extensionality and uniqueness of identity proofs. Interestingly, Hofmann's proof is "stronger" than the statement of his theorem, as the requisite language in which to speak of conservativity and equivalence of dependent type theories did not exist at the time.

In [KL18], it was observed that the category of models of a dependent type theory carries the structure of a left semi-model category. This structure was used by Isaev [Isa18] to define a *Morita equivalence* of dependent type theories as a translation between them that induces a Quillen equivalence between their categories of models.

This talk is based on our recent paper [KL25], in which we give a direct proof of Morita equivalence between the extensional type theory and the intensional type theory extended functional extensionality and UIP. While Hofmann proves that the initial models of these theories are suitably equivalent, we generalize this result to all possible extensions of the base theories by types and terms, including propositional equalities.

Therefore, thanks to proving Morita equivalence, one does not need to prove an analogue of Hofmann's result for any new extension but instead appeal to our result addressing all extensions once and for all. As new variants and extensions of intensional type theory are constantly proposed, this reduces the burden of proving their expected properties by making what should be formal formal.

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Exploring dualities beyond sound doctrines

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Abstract.

Several dualities in categorical logic (including Gabriel-Ulmer duality) admit a unified treatment in terms of the kinds of limits required to exist in the relevant "theories" (finite limits in the Gabriel-Ulmer case). Given a collection of small categories Φ (a "doctrine"), we may regard categories $\mathcal{C}$ having all limits of all shapes $\mathcal{J} \in \Phi$ as syntax-independent avatars for certain essentially algebraic theories, possibly with infinitary operations; **Set**-models of $\mathcal{C}$ are precisely the Φ-limit-preserving functors $\mathcal{C} \to \mathbf{Set}$.

A technical condition on Φ – that of being *sound* (a notion introduced in [1]) – guarantees a duality between the 2-category $\mathbf{\Phi}$**-cat** of small Φ-complete categories and that of locally Φ-presentable categories (defined much like locally finitely presentable categories, with all instances of "finite limits" swapped with "Φ-limits"). The bi-equivalence is given in one direction by taking models in **Set**, and in the other by taking (the opposite category of) Φ-presentable objects. The general theorem was proved in [2].

In this talk, we will investigate the extent to which soundness is necessary to obtain some form of duality $\Phi[-, \mathbf{Set}] \colon \mathbf{\Phi\text{-}Cat} \to \mathcal{K}^{\mathrm{op}}$. When $\mathcal{K}$ is the 2-category of locally Φ-presentable categories and the inverse 2-functor is that of taking Φ-presentable objects, it quickly follows that Φ is sound. We first drop the second requirement, showing that if $\Phi[-, \mathbf{Set}]$ is an equivalence with $\mathcal{K} = \mathbf{L\Phi P}$ then Φ must still necessarily be sound. This is done by exhibiting a (relative) right adjoint to $\Phi[-, \mathbf{Set}]$ for an arbitrary doctrine (essentially via the cartesian closedness of **Cat**). We then drop the assumption that $\mathcal{K}$ consists of the locally Φ-presentable categories. In this general situation, the existence of a duality (potentially "abstract", i.e. with $\mathcal{K}$ not defined explicitly) for Φ turns out to depend *precisely* on whether there are limits not present in Φ-complete categories that are nonetheless preserved by Φ-continuous functors (much like the existence of limits preserved by all functors). Indeed, the relevant condition on $\mathbf{\Phi\text{-}Cat}$ concerns the nature of the *lax epimorphisms* therein, which in **Cat** are shown in [3] to be precisely the absolutely (co)dense functors.

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Giry monad revisited

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Abstract.

Grasping probability by means of category theory is a longstanding problem. In 1981 Giry [1] suggested two versions of a monad structure to capture random processes—one on the general category of measurable spaces, the other on the category of Polish spaces. More recently, Tobias Fritz [2] came up with the concept of Markov categories. Unfortunately, the latter does not work together with the traditional set-theoretic way to model probability theory. And the former approach is quite weak in way of practical utility for general measurable spaces, while Polish spaces do not cover all reasonable applications.

In my contribution I want to revisit Giry's former approach. The enterprise is worthwhile, as actually a more granular understanding of the particularities of set-theoretic probability was only achieved after Giry wrote her paper—culminating for instance in [3, 4, 5].

At the outset, I am going to explain how the actual issue is the monad multiplication—based on a counter example by Ramachandran [6]. To overcome this obstacle, I, firstly, continue by visiting set theories close to ZFC. Afterwards, I will consider the Giry construction merely as an endofunctor. In this set up, I will present a new result of weak pullback preservation for the general class of countably separated measurable spaces with the Giry monad restricted to measures with quotient regular conditional probability property. Limit preservation properties (for directed limits) had already served Giry as a touchstone.

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Upgrading equivalences in a weak ω-category to coherent ones

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Abstract.

It is well known that any equivalence in a 2-category can be upgraded to an adjoint equivalence. I will present a weak ω-categorical version of this result.

More precisely, consider an ω-functor $F\colon X \to Y$ between weak ω-categories in the sense of Batanin [1] and Leinster [3]. We say F is an *ω-equifibration* if, for any n-cell x in X and for any (coinductive) equivalence $(n+1)$-cell $u\colon Fx \to y$ in Y, there exists an equivalence $(n+1)$-cell $\bar{u}\colon x \to \bar{y}$ in X such that $F\bar{u} = u$; this is an ω-dimensional analogue of isofibrations between ordinary categories. This definition cannot be directly translated into a right lifting property because it involves the *property* (as opposed to a *structure*) of u and $\bar{u}$ being equivalences. In fact the model categorical intuition tells us that, in order to characterise the ω-equifibrations by a right lifting property, we would need a notion of *coherent* equivalence. Our main result provides two such notions, namely that of *coinductive half-adjoint equivalence* and that of *coinductive equivalence with separate left and right inverses*.

We also prove that the ω-equifibrations between *strict* ω-categories are precisely the fibrations in the folk model structure constructed in [2], leading to an explicit description of generating folk trivial cofibrations (which involves the strict ω-category $\omega\mathcal{E}$ of [4]).

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Weak action representability and categories of algebras

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Abstract.

It is well known that in the semi-abelian category **Grp** of groups, internal actions are represented by automorphisms. This means that the category **Grp** is *action representable* with the actor of a group being its group of automorphisms. The notion of action representability has proven to be quite restrictive: for instance, it was proved that the only non-abelian variety of non-associative algebras which is action representable is the variety of Lie algebras. More recently G. Janelidze introduced the concept of *weakly action representable* category, which includes a wider class of categories, such as the variety of associative algebras and the variety of Leibniz algebras.

The notion of weak action representability was studied in the context of varieties of algebras: it was shown that every object X of an *algebraically coherent* variety $\mathcal{V}$ admits an *external weak representation* $\mathrm{Act}_{\mathcal{V}}(-,X) \rightarrowtail \mathrm{Hom}_{\mathbf{PAlg}}(U(-),\mathcal{E}(X))$, where $\mathcal{E}(X)$ is a partial algebra, called *external weak actor*, and U denotes the forgetful functor from $\mathcal{V}$ to the category **PAlg** of partial algebras.

The aim of this talk is to investigate the relationship between action accessibility and weak action representability in the context of varieties of algebras. Using an argument of J. R. A. Gray in the setting of groups, we prove that the varieties of k-nilpotent ($k \geq 3$) and n-solvable ($n \geq 2$) Lie algebras are not weakly action representable. This establishes that a subvariety of a (weakly) action representable variety of non-associative algebras need not be weakly action representable.

We then aim to study the representability of actions in the context of categories of *unitary* non-associative algebras, which are *ideally exact* in the sense of G. Janelidze. After describing the monadic adjunction associated with any category of unitary algebras, we prove that the categories of unitary associative algebras, unitary alternative algebras and unitary Poisson algebras are action representable. This is joint work with X. García Martínez (*University of Vigo*) and F. Piazza (*University of Messina*).

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Bicategories of lax fractions

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Abstract.

The well-known calculus of fractions of Gabriel and Zisman [1] provides a convenient way to formally invert morphisms in a category. This was extended to 2-categories by Pronk in [2]. On the other hand, in [3] the second author developed a calculus of lax fractions for order-enriched categories that formally turns a given class of morphisms into left adjoint right inverses (and provides fine control over when the Beck–Chevalley condition holds). We extend this construction to general 2-categories.

One application is a construction of the bicategory of (strict) monoidal categories and lax monoidal functors from the 2-category of strict monoidal categories and strict monoidal functors by formally adding left adjoints to the morphisms whose underlying functors are fully faithful right adjoints.

The data for the construction involves the original 2-category $\mathcal{X}$ and a collection Σ of squares, commuting up to isomorphism, whose horizontal morphisms are to become left adjoint right inverses, and which will themselves will become Beck–Chevalley squares. This collection of squares must stastify a number of axioms. The resulting bicategory of lax fractions $\mathcal{X}[\Sigma_*]$ will have the same objects as $\mathcal{X}$ and 1-cells given by cospans $A \xrightarrow{f} I \xleftarrow{s} B$ where s is a horizontal morphism from Σ. The 2-cells are certain equivalence classes of diagrams of the form

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dd, equal] & I \arrow[d, "x_1"'] \arrow[dr, phantom, "\sum^{\delta_1}"] & B \arrow[l, "r"'] \arrow[d, equal] \\
{} \arrow[r, phantom, "\overset{\alpha}{\swarrow}"] & X & B \arrow[l, "x_3"'] \arrow[d, equal] \\
A \arrow[r, "g"'] & J \arrow[u, "x_2"] \arrow[ur, phantom, "\sum^{\delta_2}"] & B, \arrow[l, "s"]
\end{tikzcd}$$

where Σ denotes that the squares come from the collection Σ.

This bicategory is universal in the sense that, for each bicategory $\mathcal{Y}$, there is an biequivalence between the bicategory of pseudofunctors from $\mathcal{X}[\Sigma_*]$ to $\mathcal{Y}$ and a bicategory whose objects are pseudofunctors from $\mathcal{X}$ to $\mathcal{Y}$ that send the horizontal morphisms of Σ to left adjoint right inverses and the squares of Σ to Beck–Chevalley squares.

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The cohomology objects of a semi-abelian variety are small

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Abstract.

A well-known, but often ignored issue in Yoneda-style definitions of cohomology objects via collections of n-step extensions (i.e., equivalence classes of exact sequences of a given length n between two given objects, usually subject to further criteria, and equipped with some algebraic structure) is, whether such a collection of extensions forms a set. We explain that in the context of a semi-abelian variety of algebras, the answer to this question is, essentially, yes: for the collection of all n-step extensions between any two objects, a set of representing extensions can be chosen, so that the collection of extensions is "small" in the sense that a bijection to a set exists.

We further consider some variations on this result, involving double extensions and crossed extensions (in the context of a semi-abelian variety), and Schreier extensions (in the category of monoids).

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Level é

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Abstract.

The dimension theory formulated in [1, Section II] does not identify dimensions with numbers but with *levels* (i.e. essential subtoposes), so that each topos determines a partial order of dimensions equipped with a 'next dimension' operator (the Aufhebung). This order need not be total and leaves room for 'extraordinary' dimensions such as infinitesimal ones [2] or as the one introduced below.

Partially inspired by the distinction between 'gros' and 'petit' toposes, [1, Section I] proposes a tentative clarification of the distinction and investigates how one class arises from the other: given an object X in a topos 'of spaces', what is the reasonable topos of pseudo-classical sheaves on X? Lawvere complains that he could not give a site-invariant description but, in order to clarify the problem, he proposes to study a class of toposes $\mathcal{E}$ such that, for every object X in $\mathcal{E}$, $\mathcal{E}/X$ has a well-defined QD-subtopos $\mathrm{P}X$. Additionally, he formulates a conjecture relating $\mathrm{P}X$ and the dimension of X.

Replacing QD toposes with étendues we can give a site-invariant description of a candidate 'petit topos' associated to each object, and a proof of the corresponding conjecture.

We say that a geometric morphism $g : \mathcal{G} \to \mathcal{S}$ is an *étendue* if there is a well-supported object G in $\mathcal{G}$ such that the composite

$$\mathcal{G}/G \longrightarrow \mathcal{G} \xrightarrow{g} \mathcal{S}$$

is localic, where $\mathcal{G}/G \to \mathcal{G}$ is the canonical geometric morphism induced by G.

Definition 1. A geometric morphism $f : \mathcal{F} \to \mathcal{S}$ has a *level é* if $\mathcal{F}$ has a largest level $\mathcal{L} \to \mathcal{F}$ such that $\mathcal{L} \to \mathcal{F} \to \mathcal{S}$ is an étendue.

We are interested in geometric morphisms $p : \mathcal{E} \to \mathcal{S}$ such that, for every object X in $\mathcal{E}$, the composite $\mathcal{E}/X \to \mathcal{E} \to \mathcal{S}$ has a level é, which will be denoted by $\acute{\mathrm{E}}X \to \mathcal{E}/X$. For such a geometric morphism p, the topos $\mathcal{E}$ is to be thought of as a topos 'of spaces' and the étendues $\acute{\mathrm{E}}X$ as the 'petit' toposes, one for each X in $\mathcal{E}$.

We will show that the topos of simplicial sets, and similar pre-cohesive presheaf toposes, are of the kind specified in the previous paragraph, describing the étendues $\acute{\mathrm{E}}X$ in very concrete terms. Time permitting, we will explain how $\acute{\mathrm{E}}X$ determines the dimension of X. (Details may be found in [3].)

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Fibrational approach to Grandis exactness for 2-categories

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Abstract.
The fundamental notion of exact sequence, from homological algebra, has been captured by numerous axiomatic frameworks in categorical algebra, among which the key framework of an abelian category. Our work contributes to the extension of the notion of exact sequence and of homological algebra to 2-dimensional category theory, with insights from the theory of 2-fibrations and that of factorization systems on a 2-category. Our investigation is motivated and guided by the study of the 2-dimensional exactness structures of the 2-category of abelian categories.

In an abelian category, the (bi)fibration of subobjects is isomorphic to the (bi)fibration of quotients. This property captures a substantial information about the exactness of a category. Indeed, as it was shown in [2], categories equipped with a proper factorization system such that the opfibration of subobjects relative to the factorization system is isomorphic to the fibration of relative quotients are precisely the Grandis exact categories. Explicitly, these are categories equipped with an ideal of null morphisms such that all kernels and cokernels relative to the ideal exist and every morphism factorizes as a cokernel followed by a kernel. Aiming at uncovering exactness structures of the 2-category of abelian categories, we develop a 2-dimensional analogue of the fibrational approach to Grandis exactness described above. More precisely, we explicitly characterize those $(1,1)$-proper factorization systems on a 2-category for which the weak 2-opfibration of relative 2-subobjects is biequivalent to the weak 2-fibration of relative 2-quotients.

As an outcome, we propose a 2-dimensional notion of Grandis exact category. Such a notion involves the concept of a 2-dimensional ideal of null morphisms and null 2-cells, which we reach via a profunctor approach. 2-dimensional ideals are a key element for the extension of the notion of exact sequence to dimension 2, as they naturally lead to the definition of 2-dimensional kernels and cokernels. Thanks to the results of [3], the 2-category of abelian categories, with suitably chosen functors called Serre functors as morphisms, is an example of our 2-dimensional notion of Grandis exact category.

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Chase—Sweedler Galois theory in additive monoidal categories

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Abstract.
Chase—Sweedler theory of Galois objects [1] was originally developed at a purely categorical level, and in two special cases, where the ambient category $\mathcal{C}$ was: (a) the category of commutative (unital) R-algebras; (b) the category of cocommutative R-coalgebras (in both cases R is a commutative unital ring). We study the intermediate level of generality, where C is the category of monoids in an additive symmetric monoidal (=tensored) category, whose tensor product functor is additive in each argument. This is a work in progress, whose main goal is to describe the Harrison functor T in terms of Ext^1 functor, as it is done in [1] for commutative R-algebras. Here T is defined in terms of Galois objects, which are essentially the same as G-torsors for suitable internal (co)groups G in $\mathcal{C}$. Apart from that we explain how to extend various auxiliary and related constructions and results from the cases of algebras and coalgebras to our context. Note that many of them involve finitely generated projective objects, and their counterparts in our context are just cartesian factors of finite cartesian powers of the unit object of the ambient monoidal category.

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Homological lemmas in a non-pointed context

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Abstract.

Homological categories, namely pointed regular protomodular categories, have been shown to constitute the good context in which the non-abelian versions of the classical homological lemmas hold. Examples of homological categories are those of groups, non-unitary rings, associative algebras, Lie algebras, topological groups and many others. In particular, for pointed regular categories, protomodularity is equivalent to the validity of the short five lemma. The nine lemma holds in every homological category, and also in every regular protomodular quasi-pointed category (meaning that the unique morphism from the initial object to the terminal one is a monomorphism). Replacing short exact sequences with exact forks, it was shown in [1] that a denormalized version of the lemma holds in regular Mal'tsev categories. In [2] the authors used the framework of star-regular categories, which is based on the notion of an ideal of morphisms, for a common description of these two versions of the nine lemma. It is shown there that, in a star-regular category with "enough trivial objects", the upper and the lower nine lemmas are equivalent. Moreover, under mild additional assumptions, the middle version of the nine lemma is equivalent to a version of the short five lemma relative to stars. These results cover the known ones concerning the homological lemmas in the pointed and quasi-pointed contexts (where $\mathcal{N}$ is the class of morphisms that factor through 0) as well as the denormalized versions of them (where $\mathcal{N}$ is the class of all morphisms).

However, this context excludes several interesting examples in which some forms of the nine lemma are valid, like unitary rings, Boolean algebras, Heyting algebras, MV-algebras and, more generally, protomodular varieties of universal algebras having more than one constant. We will introduce a categorical framework which includes all the examples just mentioned, and in which suitable forms of the homological lemmas hold. We will consider regular protomodular categories $\mathcal{C}$ equipped with a full, posetal, monocoreflective subcategory $\mathcal{Z}$ of "zero objects" such that the reflector inverts monomorphisms. Pointed and quasi-pointed regular protomodular categories are examples of our situation. Another large class of examples is given by regular protomodular categories with initial object in which the unique morphism $0 \to 1$ is a regular epimorphism. This includes, in particular, *ideally exact categories* in the sense of [3], among which there are the dual categories of all elementary toposes and all protomodular varieties of universal algebras with more than one constant. Moreover, the topological models of protomodular theories with more than one constant are examples of our situation.

This talk is supported by Shota Rustaveli National Science Foundation of Georgia (SRNSFG), through grant FR-24-9660.

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Profinite completions and clones

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Abstract.
The seminal work of Kennison and Gildenhuys [1], later studied and expanded by Leinster [2], is one of the cornerstones for the categorical approach to profinite structures and Stone duality. By characterizing the ultrafilter monad

$$\beta \quad : \quad \mathbf{Set} \longrightarrow \mathbf{Set}$$

as the codensity monad of the inclusion of finite sets into all sets, it paved the way for notions of profinite completions for various kinds of structures. In particular, the codensity monad of the inclusion of finite monoids into all monoids yields the familiar profinite completion of monoids.

Inspired by ideas coming from automata theory, we apply the codensity viewpoint to the study of clones, i.e. unisorted Lawvere theories. We define the profinite completion of clones

$$\widehat{(-)} \quad : \quad \mathbf{Clone} \longrightarrow \mathbf{Clone}$$

as the codensity monad induced by the full subcategory of clones that are pointwise finite, i.e. locally finite unisorted Lawvere theories.

Using the identification of the multisorted Lawvere theory of clones as a specific full subcategory of the free cartesian closed category on one object, we obtain two fully faithful functors

$$\mathrm{Cl}_{\mathbf{Set}} \quad : \quad \mathbf{Set} \longrightarrow \mathbf{Clone} \qquad \text{and} \qquad \mathrm{Cl}_{\mathbf{Mon}} \quad : \quad \mathbf{Mon} \longrightarrow \mathbf{Clone}$$

that encode sets and monoids as clones. A crucial, yet seemingly new fact, is that these two functors are parametric right adjoints. Together with the analog statement for finite structures, these observations are key ingredients in the proof of the two following theorems.

Theorem. For any set X, we have a clone isomorphism

$$\widehat{\mathrm{Cl}_{\mathbf{Set}}(X)} \quad \cong \quad \mathrm{Cl}_{\mathbf{Set}}(\beta X)$$

Theorem. For any monoid M, we have a clone isomorphism

$$\widehat{\mathrm{Cl}_{\mathbf{Mon}}(M)} \quad \cong \quad \mathrm{Cl}_{\mathbf{Mon}}(\widehat{M})$$

where the monoid $\widehat{M}$ is the profinite completion of M.

These two theorems demonstrate that the profinite completion of clones generalizes both the ultrafilter monad and the profinite completion of monoids.

As the elements of the free clone over a first-order signature are the trees on that signature, we call profinite trees elements of profinite completions of free clones. After having proven that profinite trees verify a strong form of parametricity, we establish a close link with the profinite λ-calculus introduced in [3].

Theorem. For any signature, the associated clones of profinite trees and of profinite λ-terms are isomorphic.

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Categorical logic meets double categories

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Abstract.

In this talk, we will discuss the relationship between *fibrations/doctrines*, which often appear in categorical logic, and *virtual double categories.* The key concept is the construction $\mathbb{B}\mathrm{il}$ of a virtual double category from a *cartesian fibration* (a Grothendieck fibration with finite products).

Double categories of relations. The double category $\mathbb{R}\mathrm{el}$ has sets, functions, and (binary) relations as objects, tight (vertical) arrows, and loose (horizontal) arrows, respectively. As relations between two sets are subsets of their product, this double category can be understood as arising from the subset fibration over **Set**. This is the archetype of our construction. Namely, one can construct from a cartesian fibration $\mathfrak{p}$ a virtual double category $\mathbb{B}\mathrm{il}(\mathfrak{p})$ whose loose arrows are *relations internal to the fibration* $\mathfrak{p}$. This is a generalization of the $\mathbb{F}\mathrm{r}$-construction in Shulman's work [3].

Why virtual? In $\mathbb{R}\mathrm{el}$, the identity relation and the composition of relations are given by $=$ and $\exists$, respectively. Since not all fibrations admit the interpretation of these logical symbols, we can only define $\mathbb{B}\mathrm{il}(\mathfrak{p})$ as a *virtual* double category, a structure that does not assume the loose identity or composition. Then, it is natural to suspect that this $\mathbb{B}\mathrm{il}(\mathfrak{p})$ is a double category if and only if the logic internal to $\mathfrak{p}$ has the logical symbols $=$ and $\exists$. The latter should be stated more precisely as the fibration $\mathfrak{p}$ being an *elementary existential fibration/doctrine.*

The main result. We will show that $\mathbb{B}\mathrm{il}(\mathfrak{p})$ is a double category with "fibrational structure" and "compatible finite products" precisely when $\mathfrak{p}$ is elementary existential. To refine this, we will present the 2-category of elementary existential fibrations as the pullback of that of cartesian fibrational double categories (a class of double categories with "double-categorical products" and "substitution") *[1] along the 2-functor* $\mathbb{B}\mathrm{il}$. We will also show that this restricted $\mathbb{B}\mathrm{il}$ (in the top row below) is locally an equivalence and explain how its essential image is characterized.

$$\begin{tikzcd}
\boxed{\text{Elementary existential fibrations}} \arrow[r, "\mathbb{B}\mathrm{il}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \boxed{\text{Cartesian fibrational double categories}} \arrow[d] \\
\boxed{\begin{array}{l}\text{Cartesian fibrations}\\ \text{(Fibrations with finite products)}\end{array}} \arrow[r, "\mathbb{B}\mathrm{il}"'] & \boxed{\text{Cartesian fibrational virtual double categories}}
\end{tikzcd}$$

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Eckmann-Hilton Arguments in Weak ω-categories

(Extended Abstract for arxiv:2501.16465)

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Abstract. We prove generalisations of the Eckmann-Hilton argument in the context of globular weak ω-categories [6]. Given a strict monoidal category, the classical Eckmann-Hilton argument [2] shows that, for any endomorphisms a, b of the monoidal unit:

$$a \circ b = a \otimes b = b \circ a$$

This argument immediately generalises to strict higher categories to show the commutativity of all compositions of cells with sufficiently degenerate boundary. We further generalise this to weak ω-categories, producing a family of equivalences witnessing these commutativity results. In this setting, the proofs become significantly more complex. In particular, $a \circ b$ and $a \otimes b$ are no longer parallel due to weak unitality. This forces us to introduce a *padding* construction, similar to those of Finster et al. [3] and Fujii et al. [5], to compare non-parallel cells. The work has been implemented in the proof assistant CATT [4] for ω-categories, and can be exported to HOTT [1].

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A categorical perspetive on the complexity of satisfying constraints

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Abstract.

The so-called algebraic approach to the *constraint satisfaction problem (CSP)* has been a prevalent method of the study of complexity of these problems since early 2000's. The core of this approach is the notion of *polymorphisms* which determine the complexity of the problem (up to log-space reductions). This theory has started with the work of Jeavons, Cohen, and Gyssens [3], and has been further developed over the past 3 decades including a generalisation of the scope to the *promise constraint satisfaction problem (PCSP)* [2]. Nevertheless, recent work also suggests that insights from other fields are immensely useful in the study of PCSPs including algebraic topology (see, e.g., [4]).

The aim of the talk is to provide a gentle introduction for category-theorists into the study of complexity of CSPs and PCSPs [1]. We show that many standard CSP notions have clear and well-known categorical counterparts. For example, the algebraic structure of polymorphisms can be described as a set-functor defined as a right Kan extension. We provide purely categorical proofs of core results of the algebraic approach including a proof that the complexity only depends on the polymorphisms. Our new proofs are substantially shorter and, from the categorical perspective, cleaner than previous proofs of the same results. Moreover, as expected, are applicable more widely. We believe that, in particular in the case of PCSPs, category theory brings insights that can help solve some of the current challenges of the field.

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Morphisms and comorphisms of sites: a double-categorical approach

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Abstract.

Geometric morphisms can be induced either from morphisms or comorphisms of sites, respectively in a contravariant and a covariant way; the first are characterized through a cover-preservation property (aside flatness), the second through a cover-lifting property. As both define a relevant notion of 1-cells between sites, one may ask whether there is a proper way to mix them altogether into a single categorical structure on sites, and if so, does it help understanding the reason for which we have those twin classes of functors rather than a single one ?

In a first part, we will explain how morphisms and comorphisms, though they do not compose with each other, can be arranged as the horizontal and vertical 1-cells of a double-category of sites. Relying on the formalism of extension-restriction applied to sieves seen as presheaves, we will then show how the sheaf-topos construction defines a horizontally contravariant, vertically covariant double-functor to the quintet double-category of topoi and discuss a few properties of this double-functor, as its relation to tabulators, or a conjoints-to-companions phenomenon.

Double-categories are a good environment to manipulate one the most expressive gadgets of category theory, the so called *exact squares* introduced by [3]: those are lax squares whose corresponding extension-restriction square is invertible; in the context of sites, one can speak more generally of *locally exact squares*, those that are sent to an invertible double cell by the sheafification double-functor. After giving an intrinsic characterization of those locally exact squares in terms of *relative cofinality* à la [1], will discuss a variety of examples in and recover several classical results of topos theory as local exactness conditions of some suited squares.

In a second part, we will discuss a reason behind this double-categorical presentation. It is known since [2] that a 2-(co)monad comes with a canonical double-category of strict (co)algebras, together with lax morphisms of (co)algebras as horizontal maps and colax morphisms as vertical maps. Here, we introduce a certain comonad sending a category to its *cofree site* containing all possible filters of sieves. Then one can exhibit sites as coalgebras for the underlying copointed endofunctor, and more crucially, cover-preserving functors as lax-morphisms of coalgebras, and cover-lifting functors as colax-morphisms of coalgebras. Modulo a few adjustments to incorporate flatness, those results give a more conceptual reason for the dichotomy between morphisms and comorphisms of sites.

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A higher categorical approach to the André-Quillen cohomology of an $(\infty, 1)$-Category

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Abstract.

Simplicial categories, that is categories enriched in simplicial sets, are a model of $(\infty, 1)$-categories. Their André-Quillen cohomology, originally introduced by Dwyer and Kan [DK2], was later re-interpreted and extended by Harpaz, Nuiten and Prasma [HNP1]. The André-Quillen cohomology of a simplicial category can be used to describe its k-invariants which in turn contain various higher homotopy information and in particular yield an obstruction theory for realizing homotopy-commutative diagrams [DK2].

Our aim is to give an algebraic, elementary and explicit approach to the André-Quillen cohomology of simplicial categories using the tools of higher category theory. For this purpose, we first observe that in order to study the n^{th} André-Quillen cohomology group of a simplicial category, it suffices to look at simplicial categories that are n-truncated, that is they are enriched in n-types. This has the advantage that we can use one of the algebraic models of n-types from higher category theory to produce an algebraic replacement for the n^{th} Postnikov truncation of a simplicial category.

We choose to use the category $\mathsf{GCat}^{\mathsf{n}}_{\mathsf{wg}}$ of groupoidal weakly globular n-fold categories arising within Paoli's model of weak n-categories [Pa3] . This category is a model of n-types with a cartesian monoidal structure. Further, every n-type can be modelled by a weakly globular n-fold groupoid, that is an object of the full subcategory $\mathsf{Gpd}^{\mathsf{n}}_{\mathsf{wg}}$ of $\mathsf{GCat}^{\mathsf{n}}_{\mathsf{wg}}$ [BP2], which is more convenient algebraically. Our model for the n^{th} Postnikov truncation of a simplicial category is a category enriched in $\mathsf{Gpd}^{\mathsf{n}}_{\mathsf{wg}}$ with respect to the cartesian monoidal structure. We call the latter an n-track category.

Using the n-fold nature of $\mathsf{Gpd}^{\mathsf{n}}_{\mathsf{wg}}$ we iteratively build a comonad on n-track categories. Using this comonad we then obtain an explicit cosimplicial abelian group model for the André-Quillen cohomology of an $(\infty, 1)$-category. This is joint work with David Blanc [BP4].

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Relational doctrines, quotient completions and projectives

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Abstract.
The notion of relational doctrine is introduce in [2] as a functorial description of the calculus of relations, very much like Lawvere's doctrines provide a functorial description of predicate logic. More specifically a relational doctrine is a contravariant functor R from the product of a category C with itself, mapping each pair (A, B) of objects of C to a poset $R(A, B)$ that is though of as the set of relations from A to B, ordered by inclusion and closed under composition, identities and the converse of a relation. Relational doctrines provide a natural setting where to deal with equivalence relations. In general relational doctrines need not have quotients and we describe a universal construction that complete a doctrine with them. We show that this construction subsumes many known examples. In particular, beside Maietti and Rosolini quotient completion of elementary and existential doctrines, we find also relational doctrines based over the category of vector and metric spaces. This last class of examples turns out to be of particular interest for the so called 'quantitative reasoning' in computer science, where one wants to interpret identity relations as distances. Since relational doctrines form a 2-category **RD** one can consider monads T over a relational doctrine R and also the corresponding Eilenberg-Moore object R^T in **RD**. We show that a relational doctrine with quotients is an instance of our completion precisely when it has enough projectives and, in this case, the doctrine of algebras R^T is the completion of its restriction to free algebras with projective generators (we compare this result with similar ones in [1, 4] and [3]).

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Posites: The Foundation of Factorization Homology

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Abstract

Factorization homology, aka factorization algebras, arises naturally in higher quantization in mathematical physics, providing an elegant categorical formalization of observables of a QFT. Despite the advanced machinery involved in building the theory, factorization algebras can be conveniently defined in the language of multicategories, thanks to the monoidal structures carried by the categories of interest.

Building on previous work of the author, we will establish that at the very core of factorization homology lies the concept of posite, a site whose underlying category is posetal, and we will revisit the theory highlighting new topos-theoretic perspectives.

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Categories of relations which compose independently

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Abstract.

Regular categories can be thought of as categories with a well behaved calculus of relations. In particular, they have pullbacks and image factorizations, which are used to compose relations.

In this work we study a number of categories which *look like* categories of relations, but which do not fit into the established theory of relations in regular categories. Examples are the category of Hilbert spaces and contractions, the category of sets and partial injective functions, and categories of probability spaces and stochastic maps, which one can think of as "probabilistic relations".

Here is roughly what happens. It is well known that the canonical product of probability spaces is not a categorical product—indeed, in general products do not exist in the categories of probability theory, and neither do pullbacks. However, products of probability spaces *do* satisfy a universal property: they are universal among those spans which make their legs *conditionally independent*, in a way that we can make precise, and which can be traced back to Simpson's work [1]. We can extend this idea from products to pullbacks (*independent pullbacks*), and use it to define a composition analogous to the one of relations. It turns out that this recovers exactly the (Markov) composition of stochastic maps [2], and the same is true for the other examples.

This work is devoted to developing a *categorical theory*, parallel to the one of regular categories, which explains these phenomena. We introduce the new notion of *epi-regular independence category*, and develop the theory of relations in these categories. In particular, we establish a strict 2-equivalence between epi-regular independence categories and *dagger categories with dilators* [3] that parallels the well-known correspondence between regular categories and tabular allegories. The dagger categories with dilators are sent to their wide subcategories of dagger-epimorphisms, and the epi-regular independence categories are sent to their associated dagger categories of relations, with composition defined through independent pullbacks and a convenient factorization.

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2-dimensional commutativity and Fox's theorem: sketchy approach

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Abstract. There is a notion of commutativity for any Lawvere theory T: as an example, the theory for commutative monoids is commutative and the one for arbitrary ones is not. On the other side of the monad-theory correspondence, we have a corresponding definition of a commutative monad which asks for a commutativity of a certain diagram. In the 2-dimensional setting, we can define a pseudocommutativity structure on a monad by wanting certain diagram to commute up to a coherent isomorphism – but the corresponding notion for Lawvere theories has not been considered.

In this talk, I am going to define a pseudocommutativity structure on a Lawvere 2-theory T using the Gray tensor product. Seeing 2-theories as enriched sketches, we can show that 2-category of T-models in Cat admits a symmetric multicategory structure which is closed. We use this observation to prove a far-reaching generalization of Fox's theorem which in its classical form says that for any symmetric monoidal category C, the coproduct in CMon(C), the category of commutative monoids in C, coincides with a tensor product in C.

A study of Kock's fat Delta

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Abstract.

The study of higher categories involves many tools based on the simplex category Δ and simplicial methods [5]. Indeed, Δ enables the encoding of coherences in geometric shapes for higher structures. However, the degeneracy maps in Δ encode the identity structure *strictly* in contrast to the associativity structure. The category referred to as fat Delta, denoted by $\underline{\Delta}$ and first introduced by Kock [4] as the category of relative finite semiordinals (i.e. relative finite ordinals with a total strict order relation), was developed as a means of providing a geometric interpretation of *weak* identity arrows in higher categories.

We present in [3] a comprehensive study of $\underline{\Delta}$ mainly via the theory of monads with arities [7, 2], which offers an abstract setting to produce nerve theorems and study Segal conditions. Our first main result is the nerve theorem for relative semicategories, denoted by $\mathsf{RelSemiCat}$.

Theorem 1 ([3, Theorem 4.23])**.** *Let* $\mathsf{RelGraph}$ *denote the category of relative directed graphs and let* $\underline{j} : \underline{\Delta}_0 \hookrightarrow \underline{\Delta}$ *be the inclusion of the wide subcategory of relative semiordinals and relative graph morphisms. The nerve functor* $\underline{\mathcal{N}} : \mathsf{RelSemiCat} \to \widehat{\underline{\Delta}}$ *is fully faithful. The essential image is spanned by the presheaves whose restriction along* $\underline{j}$ *belong to the essential image of* $\underline{\mathcal{N}}_0 : \mathsf{RelGraph} \to \widehat{\underline{\Delta}_0}$*.*

In particular, this indicates that $\underline{\Delta}$ is for relative semicategories what Δ is for categories. Among other consequences of the theory of monad with arities, we also show that $\underline{\Delta}$ has a special orthogonal factorisation system.

Proposition 1 ([3, Proposition 4.25])**.** $\underline{\Delta}$ *admits an active-inert factorisation system* $(\underline{\Delta}_a, \underline{\Delta}_0)$*.*

This active-inert factorisation system allows us to more easily express the Segal condition of [6, Section 4.3]. Additionally, using $(\underline{\Delta}_a, \underline{\Delta}_0)$, we can relate $\underline{\Delta}$ to Berger's theory [1]:

Theorem 2 ([3, Theorem 5.4])**.** $\underline{\Delta}$ *is a strongly unital and extensional hypermoment category.*

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Unified approach to pointfree T_0-spaces

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Abstract.

The classical correspondence between topological spaces and frames $\Omega\colon \mathsf{Top}^{\mathsf{op}} \leftrightarrows \mathsf{Frm}\colon \mathsf{pt}$ restricts to a dual equivalence between *sober spaces* and *spatial locales*. Thus, sober spaces are fully embedded in the category of frames, permitting us to study such topological spaces via algebraic/order-theoretical techniques.

Another correspondece that allows the study of a certain class of spaces via pointfree techniques is T_D-duality [1], which captures a correspondence between the category T_D-spaces and the category of frames with D-homomorphisms, obtained by suitably adjusting the spectrum functor. Since not every sober space is a T_D-space, this allows for a class of non-sober spaces to be studied pointfreely; however, since not every T_D-space is sober, these two approaches appear incompatible, despite being dual in some sense.

Inspired by Raney duality [2], which describes a correspondence between T_0-spaces and Raney algebras, Suarez [5, 6] developed a pointfree account of this duality via *Raney extensions* of frames, which allows for a pointfree description of all T_0-spaces, which include both sober and T_D-spaces. In fact, the spectra for both the T_D-duality and the classical correspondence appear as special cases of this pointfree Raney duality, by considering the smallest and the largest Raney extensions of a frame L.

Other pointfree notions of T_0-space are given by the *McKinsey-Tarski algebras* [3] and *strictly zero-dimensional biframes* [4], both of which contain Raney extensions as a distinguished subclass.

Aiming to understand the relationship between these three different notions of pointfree T_0-spaces, we introduce and study an abstract notion of T_0-*extensions of frames*, which are given by suitable functors $\mathcal{O}\colon \mathsf{C} \to \mathsf{Frm}$ that extend the classical correspondence $\Omega\colon \mathsf{Top}^{\mathsf{op}} \leftrightarrows \mathsf{Frm}\colon \mathsf{pt}$ to capture all T_0-spaces. By studying the fibers of $\mathcal{O}$, we are able to study various aspects of T_0-spaces abstractly.

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Pos-pretoposes and compact ordered spaces

L. Reggio
(joint work with Jérémie Marquès)

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Abstract.

A characterisation of the category KH of compact Hausdorff spaces and continuous maps was provided in [4] (see also [1] for a constructive approach). The key notion is that of filtrality: let us say that a (bounded, distributive) lattice L is *filtral* if it is isomorphic to the lattice of filters on the Boolean algebra of complemented elements of L. By extension, an object X in a category is *filtral* if its poset of subobjects is a filtral lattice. Filtral objects in KH are exactly the Stone spaces.

A weaker version of the main result of [4] can be stated as follows.

Theorem. *Let* E *be a non-trivial pretopos with terminal object* $\mathbf{1}$*. Suppose that:*

1. $\mathbf{1}$ *is a regular generator and, for every set* S*, the copower* $S \cdot \mathbf{1}$ *exists in* E*;*
2. *for every set* S*, the copower* $S \cdot \mathbf{1}$ *is filtral.*

Then E *is equivalent to* KH*.*

In a nutshell, condition 1 implies that E is equivalent to the category of algebras for the monad on Set induced by the adjunction $- \cdot \mathbf{1} \dashv \mathsf{E}(\mathbf{1}, -)$, and it follows from condition 2 that the latter is the *ultrafilter monad*. Manes' Theorem [5] then entails that E is equivalent to KH.

In this talk, I shall discuss an extension of the previous result to the category KOrd of Nachbin's compact ordered spaces and continuous monotone maps, regarded as a Pos-pretopos enriched in the category Pos of posets and monotone maps. The poset-variant of Manes' Theorem, due to Flagg [3], identifies KOrd with the category of algebras for the *prime filter monad* on Pos.

Our approach is based on the observation that a Pos-pretopos, and more generally a Pos-lex-category, can be identified with a category of internal partial orders in an ordinary lex-category. This allows us to use the internal logic to describe order-enriched variants of ordinary notions, such as filtrality. For example, in the same way that regular categories correspond to regular theories, Pos-regular categories correspond to *monotone* regular theories. Here, a theory T is monotone if for every sort X there is a binary relation symbol $\leq_X :: X \times X$ such that T proves that (i) $\leq_X$ is a partial order and (ii) every function symbol is monotone with respect to these orders.

If time permits, I will also discuss the Pos-enriched analogue of *extensive* categories and the connection with the study of two-dimensional exactness conditions in [2].

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An intrinsic approach to kernels in general categories

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Abstract. Given a nice epimorphism $p\colon A \to B$, one would hope to be able to recover it from the data of its kernel. With that goal in mind, we consider a category $\mathbb{C}$ equipped with a coreflective subcategory $\mathbb{Z}$, so that a pullback square

$$\begin{tikzcd} K \arrow[r] \arrow[d] & d(B) \arrow[d] \\ A \arrow[r, "p"] & B, \end{tikzcd}$$

where $d(B) \to B$ is the coreflection of B into $\mathbb{Z}$, exhibits the span $d(B) \leftarrow K \to A$ as the kernel of p. (Kernels in the classical sense are recovered when $\mathbb{Z}$ is the zero subcategory of a pointed category and $d(B) = 0$.) We require this pullback square to also be a pushout for every nice epimorphism p, which means asking that a nice epimorphism should be the cokernel of its kernel.

If we don't want the subcategory $\mathbb{Z}$ to be extra structure on top of $\mathbb{C}$, we can suppose that the smallest such subcategory $\mathbb{Z}$ (possibly satisfying some extra niceness conditions) exists, thus equipping the category $\mathbb{C}$ with a certain *intrinsic* notion of kernel. Additionally, under niceness assumptions, the data of the span $d(B) \leftarrow K \to A$ can be encoded into a pair (ρ, K), with ρ an equivalence relation (internal to $\mathbb{Z}$) on $d(A)$, and $K \rightarrowtail A$ a subobject. This is in essence similar to *star kernels* [1] which also tie together kernels as subobjects with kernels as equivalence relations.

The intuition is that we try to minimize the subcategory $\mathbb{Z}$, so that the ρ component of the pair (ρ, K) is as trivial as possible. For a semi-abelian category $\mathbb{C}$, since the minimal $\mathbb{Z}$ is the zero subcategory, the equivalence relation ρ contains no information, meaning the kernel of $A \to B$ is solely encoded into a subobject $K \rightarrowtail A$.

On the other hand, for $\mathbb{C} = \mathsf{Set}$, the minimal $\mathbb{Z}$ is Set itself. In this case the kernel of $p\colon A \to B$ is (ρ, A), with ρ the kernel pair of p and $A \rightarrowtail A$ always the largest subobject, therefore containing no information.

A motivating example that lies somewhere in-between is the category of inverse monoids, in which $\mathbb{Z}$ is not the zero subcategory, but instead the subcategory of semilattices, with $d(S) \subseteq S$ the semilattice of idempotent elements of the inverse monoid S. This approach then recovers the classical *kernel-trace* description [2, Section 5.1] of a surjective morphism of inverse monoids.

With $\mathbb{C}$ the category of unital rings, $\mathbb{Z}$ consists of the initial object, but note that $d(B) \to B$ and therefore $K \to A$ no longer need to be subobjects.

Finally we analyze this approach through the behavior of pullback squares of categories

$$\begin{tikzcd} \mathbb{K} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\ \mathcal{E} \arrow[r, "\mathsf{cod}"] & \mathbb{C}, \end{tikzcd}$$

pulling back a coreflective subcategory $\mathbb{Z}$ along the codomain fibration of nice epis.

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Ord-Mal'tsev categories

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Abstract.
The aim of this talk is to explore the 1-dimensional algebraic Mal'tsev property from an Ord-enriched point of view. A 1-dimensional (regular) Mal'tsev category [2, 1] may be characterised through nice properties on (internal) relations such as:

- every reflexive relation $R\colon X \looparrowright X$ is an equivalence relation;
- any relation $D\colon X \looparrowright Y$ is *difunctional*, meaning that $DD^{\circ}D \subseteq D$.

The proof of such characterisations are easily obtained through the calculus of relations, which has been well established for regular categories for several years (see [1]).

In order to explore the Mal'tsev property in an Ord-enriched context we have to develop the calculus of relations for regular Ord-categories. We adapt the calculus of relations given in [4], which was done for regular Pos-categories. To capture the enriched features of a regular Ord-category and obtain a good calculus, the relations we work with are precisely the *order ideals*. We introduce the notion of Ord-*Mal'tsev category* and show that these may be characterised through enriched versions of the above mentioned properties adapted to order ideals. Any Ord-enrichment of a 1-dimensional Mal'tsev category is necessarily an Ord-Mal'tsev category. We also give some examples of categories which are not Mal'tsev categories, but are Ord-Mal'tsev categories (see [3]).

This talk is based on joint work with Maria Manuel Clementino [3].

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Grothendieck coverages on free monoids

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Abstract.
In [1], it is shown that the continuous actions of a topological monoid (M, τ) on sets forms a coreflective subcategory of the topos[1] of *all* actions of M (discarding the topology), and as such is a topos. The latter is equivalent to the category $\mathrm{PSh}(M)$ of presheaves on M, where M is viewed as a one-object category. Given that every topos is of the form $\mathrm{Sh}(\mathcal{C}, J)$ for some category $\mathcal{C}$ and coverage[2] J on $\mathcal{C}$, a regular question in response to [1] is:

Q: What are the coverages on M and the corresponding subtoposes of $\mathrm{PSh}(M)$ like?

All but the degenerate case are *hyperconnected* over Set, which means that the resulting toposes are 'orthogonal' to the much studied motivating case of *localic* toposes over Set, built from topological spaces (or their point-free counterparts, locales). As such, these should provide a complementary class of examples.

We consider in this talk the case where $M = \Sigma^*$ is free over a set Σ. The 'sieves' constituting a coverage J on M are *right ideals*, which can be identified with upwards-closed subgraphs of the *Cayley graph* of Σ^*, a directed tree where each vertex has children indexed by Σ. These can in turn be identified with 'prefix-independent sets' and then 'full $|\Sigma|$-ary subtrees'. We use these alternative presentations to facilitate the analysis of the possible coverages. An informal statement of this classification is as follows.

Proposition. Non-trivial, non-degenerate coverages on Σ^* are indexed by sets of equivalence classes of infinite words (elements of Σ^ω). In particular, the empty set corresponds to a minimal non-trivial coverage J_{min} such that $\mathrm{Sh}(\Sigma^*, J_{min})$ is the (generalized) Jónsson-Tarski topos, and there is a maximal non-degenerate coverage J_{max} coinciding with the dense coverage.

The equivalence relation in question is that of sharing a common (infinite) suffix. This statement is informal insofar as a countable set of equivalence classes provably determines a unique coverage, but strictly more information is needed to account for the possibilities when the set is uncountable.

A more abstract approach is to consider the Lawvere-Tierney topologies on $\mathrm{PSh}(\Sigma^*)$. Conveniently, this topos is an étendue: it has a cover by a localic topos. Even more conveniently, this topos is that of sheaves on (generalized) Cantor space. As such, we arrive at a complementary understanding of these coverages in terms of sublocales of Cantor space.

We shall end the talk by explaining how these considerations can be extended to a characterization of coverages on more general monoids.

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[1] All toposes mentioned are Grothendieck toposes.
[2] We use the term coverage for what is usually called a Grothendieck topology. Given that we also consider topologies in the usual sense, this clash of terminology is best avoided.

Substitution for Substructural Theories

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Abstract. Algebraic theories as monoids for a substitution monoidal structure [4] have been considered in substructural settings, such as for linear theories [5] and affine theories [6]. We will recast these constructions noting that they arise from various free symmetric monoidal theories [1], and further generalise them to a theory encompassing linear, affine, relevant, and cartesian structures together with substructural coercions between them. Thereon, we develop a substitution monoidal structure on the corresponding presheaf category and establish that its monoids are equivalent to substructural abstract clones. This generalises the well-known equivalences of monoids for the substitution monoidal structure in the cartesian and linear settings with abstract clones and symmetric operads, respectively.

Following [2, 3], we extend the aforementioned monoidal category to a bicategory that is a model of linear logic and thus induces a cartesian closed bicategory. We similarly consider such bicategories for the various subtheories and extend each subtheory inclusion to a pseudo-adjunction. This yields various free-forgetful adjunctions between categories of theories, such as between Lawvere theories, symmetric operads, and our substructural theories.

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Stability from the categorical point of view

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Abstract.

Stability theory is a thriving modern branch of model theory, initiated by Morley [7] and largely developed by S. Shelah [8]. It aims to classify structures based upon their logical complexity and its central tool is that of independence relations, basic examples of which include linear independence in vector spaces and algebraic independence in fields. In a series of papers [3, 4, 5], we have shown that it can be viewed as categories equipped with a class of commuting squares, called independent squares. The resulting stable independence is closely related to cofibrant generations of morphisms. Later, we extended this approach to unstable independences in [1, 2].

We are going to survey this approach and to supplement it by new results. In particular, we will show how the stability spectrum is related to the small object argument. This makes it possible to treat superstability in acts over monoids (in [6] was done for modules over a ring).

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Extending strong conceptual completeness through virtual ultracategories

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Abstract.
Following Makkai's *strong conceptual completeness* [1], Lurie proved in [2] a powerful reconstruction theorem: one can recover a coherent topos (seen as a theory) as sheaves over its *ultracategory* of points (seen as models). We extend this result: every topos induces a **virtual ultracategory** structure on its points, sheaves over which yield back the original topos, assuming it has enough points.

More precisely, we show the following.

Theorem. *Topoi with enough points embed reflectively into accessible virtual ultracategories.*

Intuitively, ultracategories are categories with an ultraproduct operation, and virtual ultracategories generalize ultracategories in the same way that multicategories generalize monoidal categories. The notion of virtual ultracategory fits indeed into the framework of generalized multicategories from [3], moreover ultracategories can be recovered by adding a representability condition.

Alternatively, in the same way that ultracategories are a categorification of compact Hausdorff spaces (seen as β-algebras) one can see virtual ultracategories as a categorification of topological spaces (seen as relational β-modules). The notion of virtual ultracategory takes thus naturally its place in the missing blank below.

compact Hausdorff spaces	ultracategories	coherent topoi
topological spaces	**?**	topoi with enough points

In particular, when restricted to coherent topoi, the theorem above gives a new proof of Lurie's reconstruction theorem. The proof we present relies on representations of topoi by topological groupoids, another, more established, way to reconstruct a topos from its points.

This talk is based on [4].

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Enhancements of quivers with relations

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Abstract.
The theory of quiver representations has long played a fundamental role in both algebra and geometry. More recently, even modern fields such as topological data analysis have benefited from insights drawn from quiver representations [1].

Let k be a field. A quiver Q is a directed graph, and a representation of Q is a functor from the quiver to the category of k-vector spaces. To study the properties of the category of representations, we consider its derived category, the category of chain complexes localized at quasi-isomorphisms—i.e., morphisms inducing isomorphisms in cohomology. However, the derived category lacks a well-behaved notion of limits and colimits, as they are not functorial. To address this, we turn to enhancements of the derived category, which provide a framework for computing homotopy limits and colimits.

One such enhancement is provided by *derivators*, introduced independently by Grothendieck, Heller, and Franke, and further developed by Groth (see [4], [2]). Heuristically, a derivator can be viewed as a collection of "homotopy categories of diagrams," making it a natural tool for studying quivers. Indeed, Groth and Šťov'ıček developed a derivator-theoretic approach to the representation theory of Dynkin quivers of type A [3].

To investigate the category of representations, we can equivalently examine the module category of the path algebra of a quiver, an algebra whose underlying module is freely generated by all paths in Q, with multiplication given by concatenation. A less-explored direction in this framework is the derivator enhancement of quivers with relations, where relations are algebraic constraints imposed on the path algebra via a quotient by an ideal. A first instance of such an enhancement was studied in [5], but a general approach to extending this construction to a wider class of relations remains open.

In this talk, we address this problem by computing homotopy Kan extensions and analyzing their behavior in this context. Our results provide a derivator enhancement for all quadratic monomial relations over Dynkin quivers of type A. A direct application of this result leads to a derivator-theoretic version of derived equivalences arising from Koszul duality, showing that these equivalences are universal, meaning they do not depend on the choice of the field of coefficients.

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Mnemetic Lax Idempotent Monads and Compactness

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Abstract. Lax idempotent 2-monads (also known as *Kock-Zöberlein monads* [1, 2]) give a framework for *free cocompletions*. Examples are the *down-set monad* on posets (adjoining all joins to a poset) or the *Ind-completion* on categories (adjoining all filtered colimits). Moreover, for these monads being an algebra becomes a property making l.i. monads *property-like* [3]. The technical condition for and object $A \in \mathcal{A}$ to be an algebra of an l.i. monad $T : \mathcal{A} \to \mathcal{A}$ is that the unit $\eta_A : A \to TA$ admits a left adjoint with invertible counit.

In many of these situations, we have a notion of *compactness* or *primality*, which often allows for generators to be recovered from a cocompletion. For example, a category with finite limits can be reconstructed from its *ex-lex completion* as the full subcategory of *regular projective objects*, and any poset can be identified with the *completely join prime* elements of its suplattice completion. On the other hand, a locally small category can be recovered from its small cocompletion only up to closure under retracts (Cauchy completion), and for genuinely idempotent T there is no hope to recover A from TA.

We propose an abstract criterion to characterize l.i. monads in which A can be recovered from TA: a *mnemetic monad* is a l.i. monad for which the diagram

$$\begin{tikzcd} A \arrow[r, "\eta_A"] & TA \arrow[r, shift left=2, "T\eta_A"] \arrow[r, shift right=2, "\eta_{TA}"'] \arrow[r, phantom, "\Downarrow\theta_A"] & TTA \end{tikzcd}$$

is an *inverter* for all $A \in \mathcal{A}$, where $\theta_A : T\eta_A \to \eta_{TA}$ is the mediating 2-cell of the adjoint cylinder $T\eta_A \dashv \mu_A \dashv \eta_{TA}$.

In many cases, the adjunction arising from a l.i. monad can be decomposed into an idempotent and a mnemetic part, for example, the monadic functor $\mathsf{CoComp} \to \mathsf{CAT}$ from cocomplete to locally small categories factors through Cauchy complete categories:

$$\begin{tikzcd} \mathsf{CAT} \arrow[r, bend right] \arrow[r, phantom, "\top"] & \mathsf{CAT}_{\mathsf{cc}} \arrow[l, bend right] \arrow[r, bend right] \arrow[r, phantom, "\top"] & \mathsf{CoComp} \arrow[l, bend right] \end{tikzcd}$$

However, the naive approach to exhibit such a factorization does not work in general, the problem being that TA need not be the cocompletion of the inverter of θ_A. The talk will present a counterexample to this effect, as well as discuss notions such as *compactness* and *continuity* in the abstract framework.

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Principal bundles in join restriction categories

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Abstract. Principal bundles have three different definitions, depending on the category of geometric objects you study.

In Differential Geometry, [6] defines them as a locally trivial projection map of smooth manifolds with an atlas whose transition maps are given by group multiplication. In Topology [7] defines them as G-equivariantly trivial G-spaces. In Algebraic Geometry, [8] defines them as étale locally isotrivial geometric quotients of G-varieties.

The goal of this work is to have an overarching categorical notion that recovers all of them. While they are different objects, they all have in common that they are locally isomorphic to the Cartesian product of a base space with a group.

In order to give a purely categorical definition of a principal bundle, we formulate this condition in the language of join restriction categories. Restriction categories were developed in [1, 2, 3] to generalize partial maps (maps defined only on a subset of the domain) and have since then found applications in mathematics and computer science. Join restriction categories, as described in [5] are restriction categories where local restrictions can be used to obtain a global map. Together with a manifold construction inspired by [4] that allows us to glue together objects, this setup allows the description of principal bundles entirely in the language of join-restriction categories.

Our categorical principal bundles recover the principal bundles from topology and differential geometry and we hope to also incorporate the principal bundles from algebraic geometry as an example of this in a carefully chosen partial map category of schemes.

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A Constructive Small Object Argument

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Abstract.
The small object argument is an important tool in homotopy theory: it is commonly used to construct weak factorisation systems from collections of morphisms. It was originally proved by Quillen [3], but has evolved significantly over time. In recent work, Bourke and Garner [2] proved the most general version of the small object argument thus far, using the notion of cofibrant generation by a double category to generate algebraic weak factorisation systems.

Given a double functor $U : \mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ with $\mathbb{J}$ small, the proof reduces to showing that the induced functor $(\mathbb{J}^{\pitchfork})_1 \to \mathcal{C}^{\mathbf{2}}$ has a left adjoint. Whereas Bourke and Garner give an indirect proof, in this talk we will present a new proof of the small object argument in which we explicitly construct this adjoint. This results in a simpler and more transparent proof. One crucial ingredient that we introduce is the notion of a *one-step lifting structure*. We identify the initial object in the category of one-step lifting structures, and we utilize the induced universal property throughout our proof.

As an application, we show that a special case of our theorem yields a constructive version of the small object argument. This (constructively) shows that the effective Kan fibrations are the right class of an algebraic weak factorization system, thus resolving an open problem in the theory of effective Kan fibrations [1], which aims to develop a constructive model of homotopy type theory based on simplicial sets.

This talk is based on joint work in progress with Benno van den Berg and John Bourke.

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Partializations of Markov categories

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Abstract.

Many operations in probability theory involve constructions such as limits or integrals that are not always defined. Even a relatively innocuous construction like the average $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i$ of a sequence (X_i) of random variables is only defined when the limit exists.

A typical manner of thinking of a "partially defined morphism" $X \to Y$ is as a "totally defined" operation defined on some subobject $D \subseteq X$. In earlier works on restriction categories such as [1], this idea is formalized, leading to a construction of a category of "partial morphisms".

We develop a construction of a CD category of "partial stochastic maps" from a particular type of Markov category called a **partializable Markov category**. These generalize the span construction of [1] to a non-deterministic/non-Cartesian setting.

Explicitly, we call a Markov category $\mathcal{C}$ **partializable** when all isomorphisms are deterministic, pullbacks of deterministic monomorphisms exist and are themselves deterministic, and deterministic monomorphisms are closed under tensoring. We show that there is then a CD category $\mathsf{Partial}(\mathcal{C})$ whose objects are those of $\mathcal{C}$ and whose morphisms are isomorphism classes of spans $X \leftarrowtail D \to Y$ with $D \rightarrowtail X$ a deterministic monomorphism. Tensoring is done leg-wise.

Our main example is the category $\mathsf{BorelStoch}$ of standard Borel spaces and stochastic maps. The morphisms $X \to Y$ in $\mathsf{Partial}(\mathsf{BorelStoch})$ can be identified with stochastic maps $D \to Y$ for a measurable $D \subseteq X$, capturing the intuition of "partially defined stochastic maps".

We characterize structures in $\mathsf{Partial}(\mathcal{C})$ like the restriction partial order, determinism and split idempotents. We also show that properties such as positivity, representability (distribution objects), conditionals, and Kolmogorov products extend from $\mathcal{C}$ to its partialization.

Given distribution objects, the distribution functor P is shown to define a monad on the subcategory of deterministic morphisms, with associated **partial algebras**. We also show that the "averaging map" assigning to a distribution p on $\mathbb{R}_{\geq 0}$ its expectation $\int x\, p(dx)$ (when finite) is such a partial algebra (on standard Borel spaces).

This is companion work to [3] on categorifying the law of large numbers. There one needs "empirical sampling morphisms", intuitively taking a sequence of points and returning a sample from its empirical distribution, which need not always be defined, hence partial at best.

Works like [2] have developed similar CD categories generalizing sub-probability measures. While similar in spirit, crucial to applications such as [3] is the determinism of *all* domain injections, excluding general sub-probability measures.

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Double functorial representation of indexed monoidal structures

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Abstract.

The data of a logical doctrine acts in two directions: that of the *substitution* operation, and that of *quantification*. Double Category Theory provides the tools to capture these two actions and their relationship concomitantly: in a double category of spans, re-indexing and acting on predicates can be packaged as tight and loose arrows respectively, and in there tight arrows always have conjoints. By mapping these conjoints into a double category of quintets, we obtain adjunctions internal to a 2-category (classically, of posets). Moreover, by using the notion of adequate triple, we can study situations where the indexing category only has certain pullbacks.

Adding a monoidal structure to the fibers (and the double functor) allows us to capture both the Beck-Chevalley and Frobenius conditions, and the talk will explore this idea in more detail. We will concern ourselves with the monoidal analogues of regular hyperdoctrines and similar structures (where the objects of predicates are not necessarily posets, but rather live in some 2-category), and show how they are equivalent to (lax symmetric monoidal) double functors between spans and quintet double categories. If time permits, I will mention the connection between these ideas, Double Categorical Systems Theory, and assume-guarantee reasoning.

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The Dialectica construction for comprehension categories

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Abstract.

Function variables and induction principle for Π-types. In dependent type theory extensional Σ-types and Π-types admit a *negative formulation*, as explained in the HoTT book: the focus is on the elimination rule(s), and terms of such types can be introduced if we know how to eliminate them. The Σ-type constructor also admits a *positive presentation*, where elimination is phrased via an *induction principle* (split term): the focus is on its introduction rule, and terms of a Σ-type can be eliminated as long as *canonical* terms can be eliminated. To provide Π-types with an analogous positive presentation based on an induction principle (funsplit term), we allow *function variables* to appear in a variable context, adding to the theory a suitable context formation rule [1].

Models of function variable contexts. In this talk we consider dependent type theories with function variables, formulated as *fvccc*s. An fvccc (namely, a *comprehension category of function variable contexts*) is a split cc (*comprehension category*) whose context category has, for every display map $P_A : \Gamma.A \to \Gamma$, an *$I_A$-relative right adjoint* to the weakening functor along P_A, where I_A is the inclusion of the full subcategory of the display maps over $\Gamma.A$ into the slice category over $\Gamma.A$. This weakening of the property of *local cartesian closure* is equivalent to allowing function variables to appear in the contexts of the theory associated to such a cc.

Dialectica completion for pure dependent type theories. The ordinary free constructions [4] to add Σ-types and Π-types to a Grothendieck fibration (or a logic-enriched type theory) do not preserve the structure of a comprehension category (or a pure type theory). Therefore, we describe the two pseudomonads that add *specifically to such fvcccs* the Σ-type constructor and the Π-type constructor, respectively, exploiting their positive presentation, i.e. their *universality*. In detail, the former captures the *new* dependent types as the *old* dependent contexts of the input fvccc [2]. The latter includes the function variable contexts of the input fvccc within the dependent types of the output one. We study the distributivity law between these two pseudomonads over the 2-category of fvcccs as well as the associated *Dialectica pseudomonad*. In line with our previous work [3, 4], we provide a description of the algebras of the latter and of the theories that are obtained as Dialectica completions of other theories.

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The higher algebra of monoidal bicategories

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Abstract.
The aim of this talk is to relate precisely low-dimensional category theory and higher-dimensional category theory and to provide some applications of the latter to the former. We shall be particularly interested in the theory of monoidal bicategories, as developed by Day and Street in [1], and higher algebra, as developed by Lurie in [3]. A key difference between the two subjects is that one focuses on fully algebraic notions, while the other is essentially homotopical in nature, being part of the theory of $(\infty, 1)$-categories. Yet, as we will see, the theory of E_n-operads, which originates from the classification of n-fold loop spaces [4], provides a convenient way to handle concepts in both areas.

Our first main result shows that braided, sylleptic and symmetric monoidal bicategories, in the sense of [1], are exactly E_n-algebras, for $n = 2, 3, 4$, respectively, in the sense of [3]. The proof of this result involves introducing an $(\infty, 1)$-category of bicategories and subtle considerations on the interaction between strictification results and homotopy-coherent associative structures known as A_n-algebras.

As an application, we give a conceptual proof of the bicategorical version of the fact that monoids in a (symmetric) braided monoidal category form a (braided, and in fact symmetric) monoidal category [2]. For a monoidal bicategory $\mathcal{E}$, there is a bicategory $\mathsf{PsMon}(\mathcal{C})$ of pseudomonoids in $\mathcal{E}$ and

- if $\mathcal{E}$ is braided, then $\mathsf{PsMon}(\mathcal{E})$ is monoidal;
- if $\mathcal{E}$ is sylleptic, then $\mathsf{PsMon}(\mathcal{E})$ is braided;
- if $\mathcal{E}$ is symmetric, then $\mathsf{PsMon}(\mathcal{E})$ is sylleptic, and in fact symmetric.

This result was proved by Nick Gurski (and, independently, by Nicola Gambino) by direct calculations. Here instead, thanks to our first main result, we prove this result using the homotopy-coherent tools of higher algebra and hence largely avoid the daunting pasting diagrams that are typical of the theory of monoidal bicategories. In particular, the result follows from the Dunn Additivity Theorem for E_n-algebras proved by Lurie.

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When is $\mathsf{Cat}(\mathcal{Q})$ cartesian closed?

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Abstract.
Let $\mathcal{Q}$ be any (small) quantaloid (= category enriched in sup-lattices). Viewed as a bicategory, it serves as base for enriched categories and functors, thus producing the category $\mathsf{Cat}(\mathcal{Q})$. The category of ordered sets is an example of this construction (taking $\mathcal{Q}$ to be the one-object suspension of the two-element boolean algebra), as is the category of generalized metric spaces (taking $\mathcal{Q}$ to be the Lawvere quantale of positive real numbers). Yet these two examples behave quite differently: the first is cartesian closed, whereas the second is not [1]. This raises the question: can we find necessary and/or sufficient conditions on $\mathcal{Q}$ to have $\mathsf{Cat}(\mathcal{Q})$ cartesian closed? A result in [2] shows exactly how the exponentiability of each individual $\mathcal{Q}$-category depends on $\mathcal{Q}$. In this talk, we use this to give an elementary characterization of those quantaloids $\mathcal{Q}$ for which $\mathsf{Cat}(\mathcal{Q})$ is cartesian closed. With this characterization, we unify several known cases (previously proven using *ad hoc* methods) and we give some new examples. Based on joint work with Junche Yu [3].

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Where do ultracategories come from?

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Abstract.
Compact Hausdorff spaces are topological spaces enjoying two surprising properties: they are algebras for a monad on **Set**, namely the ultrafilter monad $\beta\colon \mathbf{Set} \to \mathbf{Set}$, whose underlying functor is the (right) Kan extension of $\mathbf{FinSet} \hookrightarrow \mathbf{Set}$ along itself. In the same spirit, ultracategories [1, 3] are categories endowed with a "topological" structure – insofar as they allow for a Stone-like duality for first-order logic – which are proved in [5] to be algebras for a pseudomonad on **CAT**. Hamad's construction resembles Rosolini's *ultracompletion* pseudomonad from his talk at CT2024, and it heavily exploits the properties of ultrafilters in its definition. However, it is still unclear how this pseudomonad relates to β: in particular, whether it is universally-induced by β.

In this talk, we will make ultracategories emerge as algebras for a pseudomonad on **CAT** induced by β by means of a (now *left*) Kan extension. The crucial observation is that ultracategories are algebras for a *relative* 2-monad on **CAT** (in the sense of [4]) over the inclusion $\mathbf{Set} \hookrightarrow \mathbf{CAT}$ of sets as discrete categories, whose underlying (2-)functor is $\beta : \mathbf{Set} \to \mathbf{CAT}$. Inspired by a similar result in [2] for relative (1-)monads, we will therefore introduce suitable conditions on a relative 2-monad T over a 2-functor $J : \mathbf{B} \to \mathbf{CAT}$ such that:

1. the left Kan extension $\tilde{T} : \mathbf{CAT} \to \mathbf{CAT}$ of T along J carries the structure of a pseudomonad;
2. T-algebras are equivalent to $\tilde{T}$-algebras.

In the case of β, this yields the following.

Theorem. Ultracategories are algebras for a pseudomonad on **CAT** whose underlying 2-functor $\tilde{\beta}\colon \mathbf{CAT} \to \mathbf{CAT}$ is the unique 2-functor such that $(\tilde{\beta}, \lambda)$ is a left Kan extension and (β, ρ) is a right Kan extension in the diagram on the right.

$$
\begin{tikzcd}
\mathbf{FinSet} \arrow[r, hook] \arrow[d, hook] & \mathbf{Set} \arrow[r, hook] & \mathbf{CAT} \\
\mathbf{Set} \arrow[ur, bend right, "\beta"] \arrow[d, hook] \arrow[from=2-1, to=1-2, phantom, "\Nwarrow\rho" near start] \arrow[from=2-1, to=1-3, phantom, "\Searrow\lambda"] & & \\
\mathbf{CAT} \arrow[uurr, bend right, "\tilde{\beta}"'] & &
\end{tikzcd}
$$

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Enriched categorical logic and accessibility

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Abstract.
In this talk I will discuss parts of the project that Jiří Rosický and myself are undertaking through the mostly unexplored field of *enriched categorical logic*. On the one hand, this is motivated by the desire of capturing under the same theory several notions of logic already developed for specific instances of enrichment: over the categories **Pos** of posets, ω-**CPO** of posets with ω-directed joins, and **Met** of metric spaces (all examples that are relevant in computer science). On the other hand, we wanted to explore applications for enrichment over the categories **Cat** of small categories, **sSet** of simplicial sets, **Ab** of Abelian groups, **DGAb** of differentially graded Abelian groups, and **Ban** of Banach spaces.

Among our goals is to define constructive and effective ways to build examples of *enriched accessible categories*; a certain kind of well-behaved enriched categories that satisfy very useful properties, but that can sometimes be difficult to recognise. Insofar, for a suitable base of enrichment $\mathcal{V}$, we have been able to exhibit specific fragments of enriched logic whose $\mathcal{V}$-categories of models describe the following subclasses of accessible $\mathcal{V}$-categories: finitary (as well as infinitary) varieties over $\mathcal{V}$ $(s=t)$[1], locally presentable $\mathcal{V}$-categories $(s=t, R(t), \wedge, \exists!)$[2], enriched injectivity classes $(s=t, R(t), \wedge, \exists)$[3], and enriched cone-injectivity classes $(s=t, R(t), \wedge, \exists, \vee)$[2].

While such presentations are in spirit very similar to those occurring in ordinary category theory, to prove the results above one needs to overcome several obstacles that are specific to the enriched setting. These, rather than the actual characterization theorems, will be the objectives of my talk. In particular, without assuming any significant background on enriched categories or logic, I will focus on the following problems: (1) what it means for an $\mathbb{L}$-structure to present a formula, (2) what rules of ordinary logic fail in this context, and (3) how existential quantification and disjunctions are interpreted over different bases (and produce interesting results).

[1] https://arxiv.org/abs/2310.11972
[2] to appear
[3] https://arxiv.org/abs/2406.12617

Extensive Morphisms

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Abstract.
Extensivity of a category can be described as a property of coproducts in the category, namely, that they are disjoint and universal. An alternative perspective is that it is a property of morphisms in a category. This talk explores this viewpoint by introducing natural notions of extensive and coextensive morphisms. A category is (co)extesnive if and only if each morphism in the category is (co)extensive. In addition, these notions allow for the study of (co)extensivity in categories in which only some of the morphisms are (co)extensive. For instance, while extensivity trivializes pointed categories, morphisms with trivial kernels in the category of pointed sets are extensive. We examine various examples of (co)extensive morphisms and discuss the implications of (co)extensivity within certain classes of morphisms.

For example, in Universal Algebra, the strict refinement property and Fraser-Horn property can be characterized by the coextensivity of product projections and of surjective homomorphisms, respectively, allowing for a categorical generalization of these concepts. Additionally, the coextensivity of all split monomorphisms in a Barr-exact category implies coextensivity of the entire category.

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Arrow algebras

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Abstract. A *locale* L is a complete poset in which the following distributive law holds:

$$a \wedge \bigvee_{b \in B} b = \bigvee_{b \in B} a \wedge b,$$

when $a \in L$ and $B \subseteq L$. Whenever you have a locale, you can obtain a topos from it by taking the category of *sheaves* over the locale: the result is called a *localic topos*. This category of sheaves over the locale L is equivalent to a category that has a description in terms of logic. Indeed, there is an equivalent category of L-sets, which are sets with an L-valued equality relation on them, where this equality relation is required to be symmetric and transitive; the morphisms of L-sets are L-valued functional relations.

The latter category can be understood as the result of a two-step process. First, one builds a *tripos* out of the locale L and then one turns this tripos into a topos by the *tripos-to-topos construction*. Importantly, there are triposes that do not arise from locales, for instance, the effective tripos, whose associated elementary topos is Hyland's effective topos, a non-localic (even non-Grothendieck) topos.

The aim of this talk is to introduce *arrow algebras* and explain the work of my former MSc students Marcus Briët and Umberto Tarantino [1, 3]. Arrow algebras are algebraic structures generalising locales. The point is that they still allow you to construct a tripos, an *arrow tripos*, and hence also an *arrow topos* by the tripos-to-topos construction. In this way arrow algebras are similar to Alexandre Miquel's *implicative algebras* [2], which they generalise.

These arrow toposes include the localic toposes, but also Hyland's effective topos. Indeed, many realizability toposes can be shown to be arrow toposes, because every *pca* (*partial combinatory algebra*) gives rise to an arrow algebra: this includes also "relative, ordered" pcas as in, for example, Jetze Zoethout's PhD thesis.

Crucially, Umberto Tarantino has developed a notion of morphism of arrow algebras which correspond to geometric morphisms between the associated triposes. This has allowed us to understand the following in purely arrow algebraic terms:

1. Every arrow morphism factors as a surjection followed by an inclusion, inducing the corresponding factorisation on the level of triposes and toposes.

2. Every subtripos of an arrow tripos coming from an arrow algebra L is induced by a *nucleus* on L. Given this nucleus, there is a simple construction of a new arrow algebra inducing the subtripos.

As a result, arrow algebras provide a flexible framework for constructing and studing new toposes.

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A Kaluzhnin–Krasner embedding theorem for non-associative algebras?

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Abstract.
For (non-abelian) groups A and B, the *Universal Embedding Theorem* of Kaluzhnin and Krasner [6] says that the *(unrestricted) wreath product* $A \wr B$ acts as a universal receptacle for any group G viewed as an extension from A to B. Recently, versions of the result were obtained for Lie algebras [7] and cocommutative Hopf algebras [1].

The aim of this talk is to report on joint work with Bo Shan Deval and Xabier García-Martínez [2], where we attempt to prove a version of the theorem for general varieties of non-associative algebras over a field. In such a variety, the objects are vector space equipped with a bilinear multiplication, eventually subject to a set of polynomial identities. By establishing a connection with the concept of *local algebraic cartesian closedness* [4], we find a *universal Kaluzhnin–Krasner embedding theorem*, valid in semi-abelian categories [5]. Via the results of [3], this allows us to fully characterise those varieties of non-associative algebras over an infinite field where the embedding theorem holds.

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The commuting tensor product of multicategories

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Abstract.
Given two algebraic theories S and T, their commuting tensor product $S \otimes T$ is a new algebraic theory in which the operations of S and T are required to commute with each other. The analogue of this operation for symmetric operads is the famous Boardman-Vogt tensor product. A unified account of these commuting tensor products was given in [4].

Here, we extend this analysis in two respects. First, we generalise it so as to make it applicable to the many-object case, recovering as special case the Boardman-Vogt tensor product of symmetric multicategories, subsuming work of Elmendorf and Mandell [2]. Secondly, we investigate how the commuting tensor product acts on bimodules, with the goal of generalising work of Dwyer and Hess on the Boardman-Vogt on bimodules between symmetric operads [1] to the setting of [5].

This work is carried out in the context of a double category equipped with an oplax monoidal structure, in the sense of [3], considering monads and monad multimorphisms therein, and involves showing that these form a representable multicategory. The application to the Boardman-Vogt tensor product arises by considering the double category of coloured symmetric sequences with the arithmetic product of [3].

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Immersions, Submersions, Local Diffeomorphisms, and Relative Cotangent Complexes in Tangent Categories

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Abstract.

The theory of tangent category theory is an area of study which exploded since it was discovered just over a decade ago (in its modern framework) in [1] by Cockett and Cruttwell. Tangent categories themselves are an abstract categorical framework to study differential-geoemtric reasoning by making use of the semantic and structural relations that the tangent bundle monad on the category **SMan** of smooth (real) manifolds encodes. This perspective has allowed for tangent categories to be used in theoretic computer science, differential geometry, applied category theory, (differential) linear logic, and even in algebraic geometry.

An important theme in tangent category theory (and various geometric flavours of mathematics) is to introduce and abstract the core structures of the morphisms which are important for differential geometry. For instance, both [3] and [2] the authors have abstracted, characterized, and studied how to incarnate submersions and local diffeomorphisms in the tangent categorical world. This begs the question how to incarnate other classes of maps such as immersions, unramified maps, and more into the general tangent category world.

In this talk based on joint work with JS Lemay (for which a preprint will appear on the arXiv between now and CT 2025), I will indicate some of the work we have done in this direction. I will introduce the tangent-categorical definitions of immersions, unramified maps, submersions, and local diffeomorphisms. Additionally, I will characterize how each class of maps incarnates in the tangent categories **SMan** of smooth manifolds, of commutative algebras $\mathbf{CAlg}_R$ over a commutative rig (semiring) R, the opposite category of commutative algebras $\mathbf{CAlg}_R^{\mathrm{op}}$ over a commutative rig R, and the category $\mathbf{Sch}_{/S}$ of schemes over a base scheme S. In particular, I will indicate that submersions (and local diffeomorphisms) in the tangent category of affine schemes involve classes of morphisms *more* general than formally smooth morphisms (and formally étale morphisms, respectively). Finally, I will also indicate how to define the relative cotangent complex in a tangent category and how it arises in many examples of interest.

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Cocompleteness of synthetic $(\infty, 1)$-categories

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Abstract. Riehl and Shulman [RS17] introduced *simplicial type theory* (*STT*) to reason about ∞-categories (meaning $(\infty, 1)$) synthetically. In our version of their theory, we work with a 0-type Δ^1 that is a bounded distributive totally ordered lattice, giving rise to all simplices Δ^n as well as their boundaries and horns. A type C is then a(n) (∞-)*category* if the induced maps $C^{\Delta^2} \to C^{\Lambda_1^2}$ and $C \to C^{\mathbb{E}}$ are equivalences (here, $\mathbb{E}$ is the free bi-invertible arrow). A type C is a(n) (∞-)*groupoid* or *space* if $C^{\Delta^1} \to C$ is an equivalence. Adding modalities from Lawvere's cohesion (discrete (co)reflection, discrete reflection, localization at Δ^1) and category theory (opposite and twisted arrow types), we gave an account to the category of spaces $\mathcal{S}$ and the universal left fibration $\pi : \mathcal{S}_* \to \mathcal{S}$, showing that $\mathcal{S}$ is directed univalent [GWB24]. Using the twisted arrow modality, we proved the Yoneda lemma for $\mathcal{S}$-valued presheaves, and developed first steps in presheaf theory and Kan extensions [GWB25]. A category C is *cocomplete* if $C \to C^I$ is a right adjoint for all small $I :_\flat \mathcal{U}$. Our main results offer simpler alternative conditions for a category to be cocomplete.

Theorem 1. *A category C is cocomplete if and only if any of the following hold:*

1. *C has finite coproducts and all sifted colimits.*
2. *C has all finite colimits and all filtered colimits.*

In the above, filtered and sifted colimits are defined using the notion of cofinal maps as introduced in STT by [GWB25] and closely follow the standard definitions from ∞-category theory: A category C is sifted if $C \to C^n$ is right cofinal for all $n : \mathbb{N}$ and filtered if $C \to C^K$ is right cofinal for all finite complexes K. We apply the above results to the category of *spectra* Sp. If (∞-)groupoids replace the category of sets in ∞-category theory, spectra take the place of abelian groups. We are now able to show that Sp is stable (Theorem 2, (2)) and use this to construct homology theories satisfying the Eilenberg–Steenrod axioms. Sp is given by the (HoTT) limit $\varprojlim(\mathcal{S}_* \xleftarrow{\Omega} \mathcal{S}_* \xleftarrow{\Omega} \ldots)$.

Lemma 2. 1. *Sp has all filtered colimits and all limits.*
2. *Sp is finitely (co)complete, $\mathbf{0}_{\mathsf{Sp}} \cong \mathbf{1}_{\mathsf{Sp}}$, and pushouts and pullbacks coincide.*

Corollary 3. *Sp is cocomplete.*

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Metric spaces, entropic spaces and convexity

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Abstract.

A certain notion of convexity of sets can be captured by a monad, known as a convexity monad or barycentric monad; this is a finite version of so-called probability monads. Various authors (including Mardare, Panangaden and Plotkin [4], and Fritz and Perrone [2]) have looked at convexity/probability monads on categories of metric spaces. The work of Fritz and Perrone can be recast in terms of enriched categories if you consider metric spaces as $\overline{\mathbb{R}}_+$-categories, that is, as categories enriched over the quantale of extended non-negative real numbers $\overline{\mathbb{R}}_+$.

One can then do a similar thing for any 'suitably convex' quantale R and define a convexity monad on the category of R-categories. In particular, if we consider the quantale $\overline{\mathbb{R}}$, the extended real line $[-\infty, \infty]$ with the opposite order to that used in metric spaces, then $\overline{\mathbb{R}}$-categories are what Lawvere [3] called 'entropic spaces' and argued gave a necessary structure for state spaces in thermodynamics. The category of strict algebras with lax algebra maps for the convexity monad in this case is the category of convex entropic spaces with concave maps. The hope is that this connects Lawvere entropic approach to thermodynamics with the approach of Baez, Lynch and Moeller [1] which involves convex spaces and concave maps.

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Quasi-homeomorphisms of topological groupoids

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Abstract.
Topological spaces. In the very early days of topos theory, it was noted that two (T_0-)topological spaces X, Y have equivalent topoi of sheaves $\mathbf{Sh}(X) \simeq \mathbf{Sh}(Y)$ if and only if X and Y admit subspace inclusions $X \subseteq W \supseteq Y$ that are moreover *quasi-homeomorphisms* [5, §10]. As a result, invariants of the topos of sheaves on a space (e.g. *sheaf cohomology*) correspond to invariants of the space that are preserved by quasi-homeomorphism. In some ways, this is a vestige of point-set topology – the topos of sheaves on a space X is entirely determined by the locale of opens $\mathcal{O}(X)$.

Topological groups and groupoids. In contrast, topological groups and topological groupoids do not necessarily yield localic groups or localic groupoids, and so we cannot rely on the arguably better-behaved point-free setting (as found in [6, §7]). Indeed, the question of when two topological groups G, G' are *Morita equivalent*, i.e. their topoi of continuous actions are equivalent $\mathbf{B}G \simeq \mathbf{B}G'$, becomes more complex than the point-free variant [7] (cf. also the difference between topological and localic representation theorems [3, 4]).

Our contribution. In this talk, we discuss the case for topoi of sheaves on topological groupoids.

1. We introduce the subclass of *logical* groupoids (so named for the *logical topologies* of [1]). These generalise those T_0-topological groups whose open subgroups generate the topology, or equivalently the topological subgroups of a symmetric group $\Omega(X)$ endowed with the pointwise convergence topology. The class of logical groupoids is not restrictive since every topos with enough points is the topos of sheaves on a logical groupoid (see [2]).
2. We characterise which inclusions of logical groupoids $\mathbb{Y} \subseteq \mathbb{X}$ yield an equivalence of their sheaf topoi $\mathbf{Sh}(\mathbb{Y}) \simeq \mathbf{Sh}(\mathbb{Y})$ ([8, Theorem 2]) – the titular 'quasi-homeomorphisms' of topological groupoids.

As a consequence, we will deduce that two logical groupoids $\mathbb{X}, \mathbb{Y}$ are Morita equivalent if and only if they can be embedded into a common logical groupoid $\mathbb{X} \subseteq \mathbb{W} \supseteq \mathbb{Y}$ via 'quasi-homeomorphisms' ([8, Corollary 6.3]).

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Locales are dense in Toposes

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Abstract.
It is by now a well-established fact that Grothendieck's generalised spaces, *toposes*, can be represented by their lower-dimensional analogues, *locales*. Indeed, in their seminal paper [2], Joyal and Tierney showed that every topos can be expressed in terms of a localic groupoid, and this line of work was then taken up by many other authors.

These results tell us that each topos $\mathcal{E}$ can be built out of a well-chosen finite amount of localic data. But one may wonder, what if one took a more global point of view and tried to recover $\mathcal{E}$ from all of its *canonical* localic data? That is to say, is the subcategory of locales *dense* in the category of toposes?

In this talk, we answer this question in the affirmative by showing that every topos $\mathcal{E}$ can be recovered from its "functor of *localic points*" $\mathrm{LPts}(\mathcal{E})$, which is a geometric *stack* over the category of locales as shown in [1]. The category $\mathrm{LPts}(\mathcal{E})_X$, for a locale X, is that of geometric morphisms $\mathrm{Sh}(X) \to \mathcal{E}$. In other words, $\mathrm{LPts}(\mathcal{E})$ consists of all the generalised points of $\mathcal{E}$ with localic domain.

Precisely, we show that the 2-category of toposes embeds fully faithfully via LPts in the 2-category of stacks over locales.

$$\mathrm{Topos} \xhookrightarrow{\mathrm{LPts}} \mathrm{Stack}(\mathrm{Loc})$$

This answers the question of density as LPts is exactly the nerve of the inclusion of locales in toposes. The proof relies on the stackiness of $\mathrm{LPts}(\mathcal{E})$, the description of stackification in [1], and the covering theorem of [2].

As a consequence of this result, one may now define a notion on locales and extend it to toposes "by continuity" in a canonical fashion. Time allowing, we will illustrate this principle.

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Infinite and Non-Rigid Reconstruction Theory

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Abstract.
Monadic reconstruction theory—relating additional structure of a monad to structure on its Eilenberg–Moore category—can be seen as a generalisation of classical Tannaka–Krein duality, in which one reconstructs a compact group from its category of representations. Results of this kind were obtained for oplax monoidal monads—also called bimonads—by Moerdijk [1] and McCrudden [2]; for Hopf monads by Bruguières, Lack, and Virelizier [4, 5]; and for $*$-autonomous and linearly distributive monads by Pastro and Street [6, 7]. Crucial in all of these statements is the involvement of a fibre functor, which generalises the, classically, forgetful strict monoidal functor to the category of vector spaces or bimodules over a ring.

A different kind of reconstruction is possible if one foregoes such a fibre functor. For example, given a monoidal category $\mathcal{C}$, one could ask when a given $\mathcal{C}$-module category $\mathcal{M}$ is equivalent to the Eilenberg–Moore category of some monad on $\mathcal{C}$. That is, we only recover the algebraic object of interest up to Morita equivalence.

This talk generalises such a reconstruction result by Ostrik [3] about Hopf algebras on finite tensor categories to the general case of right exact lax module monads on a nice module category over a general abelian monoidal category with enough projectives. Crucially, the proof does not need any rigidity assumptions on the underlying category, and in fact leads to a characterisation of right exact lax module monads up to Morita equivalence.

As an application, we give conceptual proofs of the fundamental theorem of Hopf modules, and the fact that a bimonad is Hopf if and only if it is strong as a module monad over its base category. The talk is based on joint work with Mateusz Stroiński [8].

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