$(\infty, 2)$-Topoi and descent.

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Introduction

Theory of $(\infty, 1)$-topoi

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Theory of $(\infty, 1)$-topoi $\longrightarrow$

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Introduction: $(\infty, 1)$-topoi

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- $\mathcal{X}$ is locally cartesian closed and admits classifiers (for large κ).

Q: How can we categorify this? . . .

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What is an $(\infty, 2)$-topos?

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Synthetic theory of fibrations

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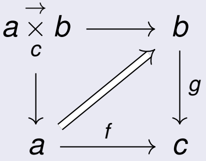

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$$\begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & & \downarrow \\ \mathrm{Free}^0_c(p) & \longrightarrow & \mathrm{Free}^0_c(q) \end{array} \longrightarrow \textbf{adjointable squares}$$

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- 0-fibration=**cocartesian fibration**.

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restriction

ϵ-fibrations, $p\colon x \to c$

E-(op)lax families of ϵ-fibrations,
$p_i\colon x \times_c c_i \to c_i, \quad i \in \mathbb{I}$

taking (op)lax colimits

Fibrational descent

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Suppose $\mathbb{X} \longrightarrow$

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Fibrations are exponentiable

The main theorem

Theorem [A-Martini 24]

Let $\mathbb{X}$ be a presentable $(\infty, 2)$-category.

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which **preserves oriented pullbacks and the terminal object**.

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$$\mathrm{Sh}(\mathcal{X}) \longrightarrow F\colon \mathcal{X}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_{(\infty,1)} \textbf{ limit preserving}$$

Theorem [A-Martini 24]: Localic reflection

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Internal (higher) category theory

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Internal (higher) category theory following **Martini-Wolf** embedds

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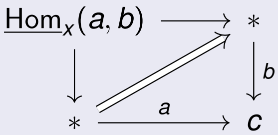

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$$\mathbb{X}(x, y)^{\simeq} \xrightarrow{\simeq} \underline{\mathrm{Hom}}_{\Omega^0}(x, y)$$

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- **Not enough groupoids**

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- (Lax) **(co)limits**

Slightlty more exotic than the $(\infty, 1)$-localic case

- **Not enough groupoids**
- In general

What can we do with this?

Synthetic category theory

- **Yoneda's lemma**
- (Lax) **Kan extensions**
- (Lax) **(co)limits**

Slightlty more exotic than the $(\infty, 1)$-localic case

- **Not enough groupoids**
- In general **no op's**

Future directions

- **Categorified sheaves and lax descent**

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- **Categorified sheaves and lax descent**
- **The enveloping** $(\infty, 2)$-topos of $(\infty, 1)$-topoi
- **Stacky formulation** for stratified $(\infty, 1)$-topoi.

Thank you for listening!