STRONGLY FINITARY METRIC MONADS ARE TOO STRONG

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Category Theory CT 2005
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Two Related Open Problems

Met : metric spaces (distance ∞ allowed)
nonexpanding maps

Problem 1 Are strongly finitary endofunctors on Met closed under composition?

Problem 2 Are varieties of finitary quantitative algebras precisely the Eilenberg-Moore categories Met^T, T strongly finitary?

Asked since 2022 by Parker, Lucyshyn-Wright, Rosický, J.A. ...

Strongly Finitary Functors

$$\mathrm{Met}_f \overset{K}{\hookrightarrow} \mathrm{Met} \circlearrowleft T$$

T finitary: preserves directed colimits

$$\Updownarrow$$

$$T = \mathrm{Lan}_K (TK)$$

T strongly finitary: $T = \mathrm{Lan}_J (TJ)$

$$\mathrm{Set}_f \overset{J}{\hookrightarrow} \mathrm{Met} \circlearrowleft T$$

Examples: $X \mapsto X^n = [n, X]$ strongly fin. ✓

$X \mapsto [P, X]$, P finite

finitary ✓

strongly $\Leftrightarrow$ P discrete

Kelly & Lack 1993

Varieties of finitary algebras over

$\mathcal{V} =$ Set, Pos, UMet...

Thm: Varieties are precisely

$$\mathcal{V}^T$$

for strongly finitary monads T.

Set, Pos, UMet ... cartesian closed

$\Rightarrow$ strongly finitary functors compose

(Kelly & Lack)

Met is symmetric monoidal closed

$X \otimes Y$ addition metric

$$d((x,y),(x',y')) = d(x,x') + d(y,y')$$

$X \times Y$ maximum metric

$$d((x,y),(x',y')) = d(x,x') \vee d(y,y')$$

Formula $T = \mathrm{Lan}_J (TJ)$: in the enriched sense

Quantitative Algebras

Mardare, Panangaden & Plotkin

since 2016

$\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$ a signature

Objects: A, a metric space

$\sigma : A^n \to A$ nonexpanding $\forall \sigma \in \Sigma_n$

with respect to $\times$, not $\otimes$

Morphisms: nonexpanding homomorphisms

Example : MONOIDS

Quantitative

$$\boxed{\begin{array}{c}\text{free on } X\text{:}\\ TX = \coprod_{n\in\mathbb{N}} X^n\end{array}}$$

strongly finitary

In the monoidal category

$$\boxed{\begin{array}{c}\text{free on } X\text{:}\\ TX = \coprod_{n\in\mathbb{N}} X^{\otimes n}\end{array}}$$

not even enriched

Quantitative Equations

$$t =_\varepsilon t' \quad (\varepsilon \geq 0)$$

t, t' terms in $T_\Sigma V$, free on a set V

<u>Definition</u> A <u>variety</u> is a full subcategory of quantitative algebras, specified by a set of quantitative equations.

<u>Example</u> (1) Quantitative monoids

(2) 1-commutative monoids:

$$xy =_1 yx$$

<u>Theorem</u> $\mathcal{V}$ a variety

(1) Free algebras exist:

$U : \mathcal{V} \to \mathbf{Met}$, a right adjoint

... monad $T_\mathcal{V}$ on Met

(2) $\mathcal{V} \simeq \mathbf{Met}^{T_\mathcal{V}}$.

Earlier Work joint with M. Dostál & J. Velebil

① Every strongly finitary monad is $T_{\mathcal{V}}$ for some variety $\mathcal{V}$.

② If strongly finitary functors compose: $T_{\mathcal{V}}$ is always strongly finitary.

Without any assumption:

$$\boxed{\text{In } \mathsf{Mnd}_f(\mathsf{Met}) \ldots \text{ finitary enriched monads}}$$

$T_{\mathcal{V}}$ a colimit of strongly finitary monads.

A Counter-Example

Binary operations $\Box, \Diamond$

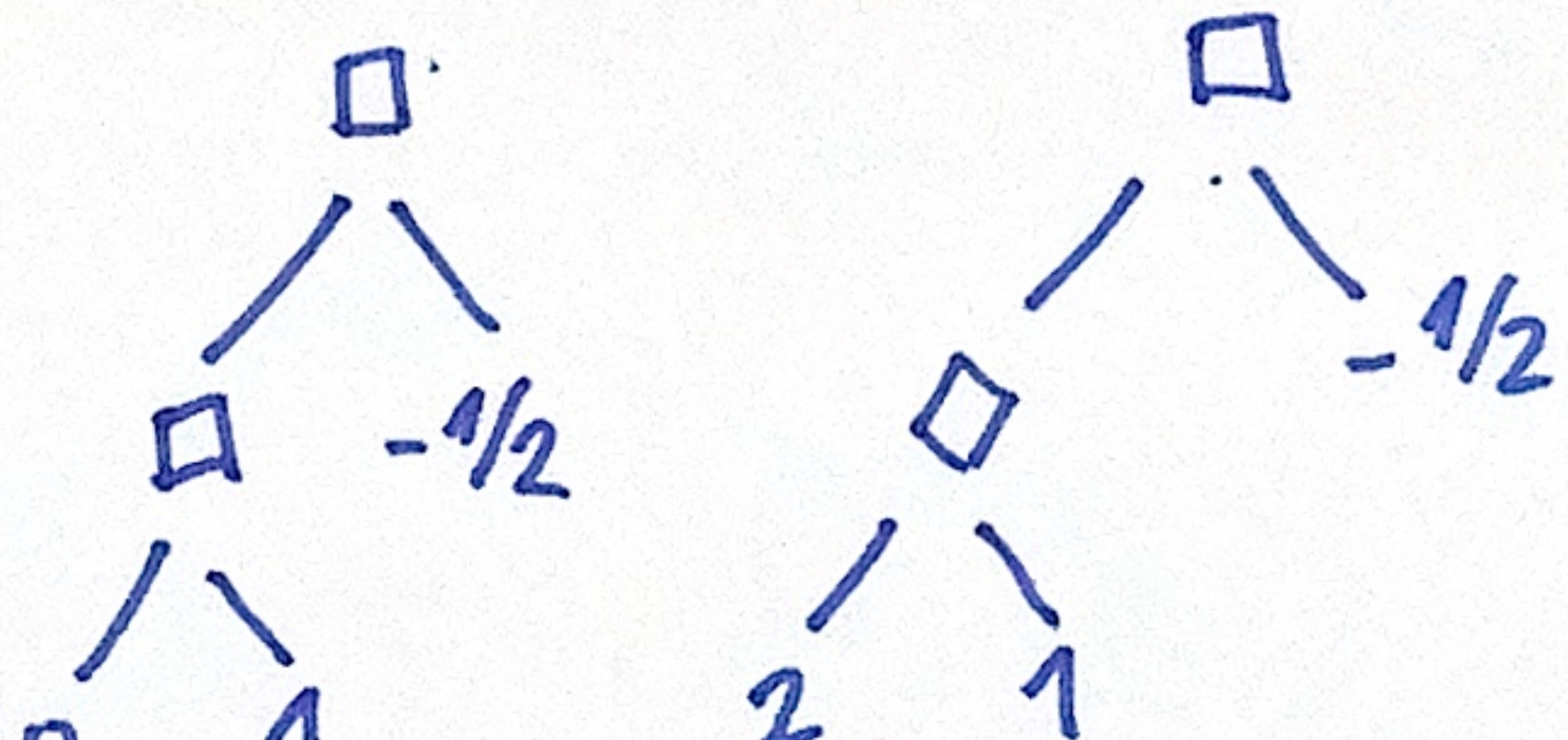
Equation $x \Box y =_\varepsilon x \Diamond y \quad 0<\varepsilon<1$

$T_\nu X$: all binary trees labelled by

X ... on the leaves

$\Box$ or $\Diamond$ ... elsewhere

Example: $T_\nu \mathbb{R}$, $\mathbb{R}$ the usual metric



distance $2+\varepsilon$

Theorem The monad T_ν is not strongly finitary.

Corollary strongly finitary functors do not compose.

The Main Theorem

The monads $T_{\mathcal{V}}$, $\mathcal{V}$ a variety, form the closure of all strongly finitary monads under weighted colimits in $\mathrm{Mnd}_f(\mathrm{Met})$.

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References

A [1] J. Adámek, *Varieties of quantitative algebras and their monads*, Proc. Logic in Computer Science (LICS 2022), 1-12.

ADV [2] J. Adámek, M. Dostál and J. Velebil, *Quantitative algebras and a classification of metric monads*, arXiv:2210.01565.

ADVP [3] J. Adámek, M. Dostál and J. Velebil, *A categorical view of varieties of ordered algebras*, Math. Structures Comput. Sci. **32**(2022), 1–25.

AMMU [4] J. Adámek,, S,. Milius, L. Moss, H. Urbat, *On finitary functors and their presentations*, J. Comput. System sci. **81** (2015), 813–833.

AMV [5] J. Adámek, S. Milius and J. Velebil, *Some remarks on finitary and iterative monads*, Appl. Categ. Struct. 11 (2003), 521–541.

AR [6] J. Adámek and J. Rosický, *Approximate injectivity and smallness in metric-enriched categories*, J. Pure Appl. Algebra **226**(2022), 1–30.

BW [7] M. Barr and C. Wells, *Toposes, triples and theories*, Springer, New York 1985.

BHMW [8] F. van Breugel, C. Hermida, M. Makkai and J. Worrell, *Recursively defined metric spaces without contraction*, Theoret. Comput. Sci. **380**(2007), 143–163.

K [9] M. Kelly, *Structures defined by finite limits in the enriched context, 1*, Cahiers Topol. Geom. Diff. **33** (1982), 3–42.

KL [10] G. M. Kelly and S. Lack, *Finite-product-preserving-functor, Kan extensions and strongly finitary 2-monads*, Appl. Categ. Structures **1**(1993), 85–94.

KV [11] A. Kurz and J. Velebil, *Quasivarieties and varieties of ordered algebras*, Math. Struct. Comput. Sci. 27 (2016), 1–42.

LP [12] R. B. B. Lucyshyn-Wright and J. Parker, *Enriched structure-semantics adjunctions and monad theory equivalences for subcategories of arities*, Theory Appl. Categories **41** (2024), 1873–1918.

ML [13] S. Mac Lane, *Categories for the working mathematician*, 2nd edition, Springer 1968.

MPP16 [14] R. Mardare, P. Panaganden and G. D. Plotkin, *Quantitative algebraic reasoning*, Proc. Logic in Computer Science (LICS 2016), IEEE Computer Science 2016, 700–709.

MPP17 [15] R. Mardare, P. Panaganden and G. D. Plotkin, *On the axiomatizability of quantitative algebras*, Proc. Logic in Computer Science (LICS 2017), IEEE Computer Science 2017, .

MU [16] S. Milius and H. Urbat, *Equational axiomatization of algebras with structure*, arXiv 1812-02016v2.

M [17] E. Moggi, *Notions of computation and monads*, Inform. and Comput. **93**(1991), 55–92.

R [18] J. Rosický, *Metric monads*, Math. Struct. Comput. Sci. **31**(5) 2021, 535–552.