Fundamental Properties of Monads in Double Categories

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CT 2025
Masaryk University, Brno

14 July 2025

GENERAL SECRETARIAT FOR RESEARCH AND TECHNOLOGY

H.F.R.I. Hellenic Foundation for Research & Innovation

Outline

1. Double categories and examples
2. Monads in double categories
3. Basic categorical properties

Double categories

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- A (pseudo-)*double category* $\mathbb{D}$ consists of
  - 0-cells & vertical 1-cells which form a category $\mathbb{D}_0$
  - horizontal 1-cells & 2-cells which form a category $\mathbb{D}_1$
  - functor $1 \colon \mathbb{D}_0 \to \mathbb{D}_1$ providing units
  - functors $s, t \colon \mathbb{D}_1 \to \mathbb{D}_0$ providing source and target

$$\begin{array}{ccc} X & \xrightarrow{A} & Y \\ {\scriptstyle f}\downarrow & \Downarrow\alpha & \downarrow{\scriptstyle g} \\ Z & \xrightarrow[B]{} & W \end{array}$$

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together with natural $(A \odot B) \odot C \cong A \odot (B \odot C)$, $A \odot 1_X \cong A \cong 1_Y \odot A$ with identity vertical boundaries, satisfying coherence axioms.

E.g. the two kinds of compositions of 2-cells obey interchange law

$$\begin{array}{ccccc} & \rightarrow\!\!\!\!\!+ & & \rightarrow\!\!\!\!\!+ & \\ \downarrow & \Downarrow\alpha & \downarrow & \Downarrow\beta & \downarrow \\ & \rightarrow\!\!\!\!\!+ & & \rightarrow\!\!\!\!\!+ & \\ \downarrow & \Downarrow\gamma & \downarrow & \Downarrow\delta & \downarrow \\ & \rightarrow\!\!\!\!\!+ & & \rightarrow\!\!\!\!\!+ & \end{array} \qquad (\delta \odot \gamma) \cdot (\beta \odot \alpha) = (\delta \cdot \beta) \odot (\gamma \cdot \alpha)$$

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$\star$ Alternative approach to 2-dimensional category theory, often more rich: for objects (0-cells) of interest, two different kinds of morphisms (with strict vs pseudo associative composition) encompassed in single structure.

Examples of double categories

- $\mathbb{R}$el with sets as 0-cells, functions as vertical 1-cells ($\mathbb{R}\mathrm{el}_0$=Set), relations $A \subseteq X \times Y$ as horizontal 1-cells $A\colon X \nrightarrow Y$, maps of relations $(xAy \Rightarrow f(x)Bg(y))$ as 2-cells.

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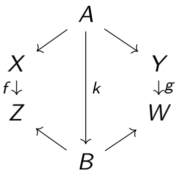
- $\mathbb{S}$pan with $\mathbb{S}\text{pan}_0$=Set, horizontal 1-cells spans $X \leftarrow A \rightarrow Y$ and 2-cells



Horizontal composition given by taking pullbacks of spans.

Works in any $\mathcal{C}$ with pullbacks$\rightsquigarrow$ double category $\mathbb{S}\text{pan}(\mathcal{C})$.

- $\mathbb{B}\mathrm{im}$ with $\mathbb{B}\mathrm{im}_0 = \mathrm{Ring}$, the category of rings and ring homomorphisms, horizontall 1-cells $R \overset{M}{\nrightarrow} S$ are (S, R)-bimodules and 2-cells $\begin{array}{ccc} R & \overset{M}{\nrightarrow} & S \\ {\scriptstyle f}\downarrow & \Downarrow\phi & \downarrow{\scriptstyle g} \\ R' & \underset{M'}{\nrightarrow} & S' \end{array}$ homomorphisms $\phi\colon M \to M'$ s.t. $\phi(mr) = \phi(m)f(r), \phi(sm) = g(s)\phi(m)$. Horizontal composition $R \overset{M}{\nrightarrow} S \overset{N}{\nrightarrow} T$ is tensor product $N \otimes_S M$.

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- $\mathcal{V}$-$\mathbb{M}\mathrm{at}$ for $(\mathcal{V}, \otimes, I)$ monoidal category+assumptions. $\mathcal{V}$-$\mathbb{M}\mathrm{at}_0 = \mathsf{Set}$,

$X \overset{A}{\nrightarrow} Y$ are $\mathcal{V}$-matrices $Y \times X \xrightarrow{A} \mathcal{V}$ i.e. $\{A(y,x)\}_{y,x}$ in $\mathcal{V}$, 2-cells are

$$\begin{array}{ccccc} Y \times X & & \overset{A}{\longrightarrow} & & \mathcal{V} \\ & {\scriptstyle g\times f}\searrow & \Downarrow\alpha & \nearrow{\scriptstyle B} & \\ & & W \times Z & & \end{array} \qquad \alpha_{x,y} \colon A(y,x) \to B(gy, fx) \in \mathcal{V}$$

Composition is 'matrix multiplication' $(B \odot A)(z,x) = \sum_y B(z,y) \otimes A(y,x)$.

Fibrant structure

- $\mathbb{D}$ is *fibrant* (or a *framed bicategory*) when the functor $(s, t)\colon \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$ is a fibration.

$F\colon \mathcal{C} \to \mathcal{X}$ is a fibration when for every $f\colon X \to F(B)$ in $\mathcal{X}$ there exists unique lifting $f^*(B) \to B$ of f in $\mathcal{C}$ with factorization property.

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Gives canonical way to turn vertical 1-cells $X \xrightarrow{f} Y$ to two horizontal ones, the *companion* $\hat{f}\colon X \to\!\!\!\!\!\!\bullet\;\; Y$ & the *conjoint* $\check{f}\colon Y \to\!\!\!\!\!\!\bullet\;\; X$.

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- In $\mathcal{V}$-$\mathbb{M}$at, $X \xrightarrow{f} Y$ gives matrices $\hat{f}(x,y) = \check{f}(y,x) = \begin{cases} I \text{ if } fx = y \\ 0 \text{ if } fx \neq y \end{cases}$

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- In $\mathbb{B}\mathrm{im}$, given $f\colon R \to S$, $\hat{f}$ is the canonical bimodule ${}_S S_R$ (restriction of scalars on the right) and $\check{f}$ is ${}_R S_S$.

Monads in double categories

- A monad in $\mathbb{D}$ is a horizontal 1-cell $A\colon X \nrightarrow X$ with 'multiplication' and 'unit' 2-cells

$$
\begin{array}{ccccc}
X & \xrightarrow{A} & X & \xrightarrow{A} & X \\
\| & & \Downarrow\mu & & \| \\
X & & \xrightarrow[A]{\qquad\qquad} & & X
\end{array}
\qquad
\begin{array}{ccc}
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satisfying usual associativity and unitality axioms. E.g.

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$\star$ Since all 2-cells are globular, coincide with monads in *bicategories*. However, maps of monads are different!

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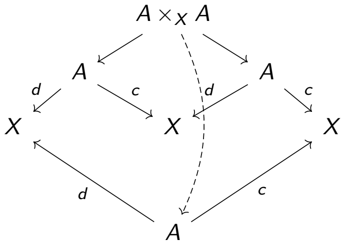
■ $\mathsf{Mnd}(\mathbb{D}) \to \mathbb{D}_0$ is a fibration.

For a vertical $X \xrightarrow{f} Y$ and monad $Y \overset{A}{\nrightarrow} Y$, $f^*(A)\colon X \overset{\hat{f}}{\nrightarrow} Y \overset{A}{\nrightarrow} Y \overset{\check{f}}{\nrightarrow} X$.

Examples of categories of monads

- For $\mathbb{D}=\mathbb{S}\mathrm{pan}(\mathcal{C})$, a monad $X \xleftarrow{d} A \xrightarrow{c} X$ is a category *internal* to $\mathcal{C}$: consists of object X of objects, object A of arrows, η picks identities and μ



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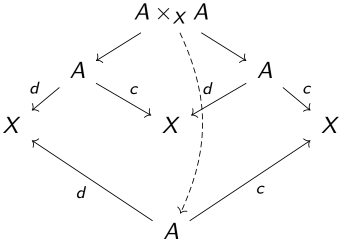

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- For $\mathbb{R}\mathrm{el}(\mathcal{C})$, category of monads $\mathsf{Mnd}(\mathbb{R}\mathrm{el}(\mathcal{C}))$ is $\mathsf{Preord}(\mathcal{C})$, category of internal preorders and order-preserving maps in $\mathcal{C}$.

- For $\mathbb{B}$im, a monad $R \overset{A}{\to\!\!\!\!+} R$ is an R-algebra and a monad map $\begin{array}{ccc} R & \overset{A}{\to\!\!\!\!+} & R \\ {\scriptstyle f}\downarrow & \Downarrow\alpha & \downarrow{\scriptstyle f} \\ S & \underset{B}{\to\!\!\!\!+} & S \end{array}$

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$\star$ Both internal and enriched categories can be studied in this context!

Parallel limits and colimits

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- $\mathbb{S}\text{pan}(\mathcal{C})$ has all parallel limits that $\mathcal{C}$ has.
- $\mathcal{V}$-$\mathbb{M}$at has parallel coproducts, and is parallel cocomplete when $\mathcal{V}$ is and $\otimes$ preserves colimits.
- $\mathbb{B}$im is parallel cocomplete.

Parallel limits and fibers

Proposition

Suppose $\mathbb{D}$ is a fibrant double category such that $\mathbb{D}_0$ is complete. The following are equivalent:

1. $\mathbb{D}$ is parallel complete;
2. The fibrations $\mathfrak{s}, \mathfrak{t} \colon \mathbb{D}_1 \to \mathbb{D}_0$ have all fibred limits: ${}^X\mathbb{D}_1$ and $\mathbb{D}_1^Z$ are complete categories for any $X, Z \in \mathbb{D}_0$, and $- \odot \hat{f} \colon {}^Y\mathbb{D}_1 \to {}^X\mathbb{D}_1$ and $\check{g} \odot - \colon \mathbb{D}_1^W \to \mathbb{D}_1^Z$ are continuous functors for any $f \colon X \to Y$ and $g \colon Z \to W$;
3. $\mathcal{H}(\mathbb{D})(X, Z)$ is a complete category for any $X, Z \in \mathbb{D}_0$, and $- \odot \hat{f} \colon \mathcal{H}(\mathbb{D})(Y, Z) \to \mathcal{H}(\mathbb{D})(X, Z)$ and $\check{g} \odot - \colon \mathcal{H}(\mathbb{D})(X, W) \to \mathcal{H}(\mathbb{D})(X, Z)$ are continuous functors.

The endomorphism category

Given a double category $\mathbb{D}$, can form the category $\mathrm{End}(\mathbb{D})$ which has:

- Objects: $A \colon X \nrightarrow X$.
- Morphisms: $\begin{array}{ccc} X & \overset{A}{\nrightarrow} & X \\ {\scriptstyle f}\downarrow & \Downarrow\alpha & \downarrow{\scriptstyle f} \\ Y & \underset{B}{\nrightarrow} & Y \end{array}$

- e.g. for $\mathbb{D} = \mathcal{V}\text{-}\mathbb{M}\mathrm{at}$, $\mathrm{End}(\mathbb{D}) = \mathcal{V}\text{-}\mathrm{Grph}$

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Given a double category $\mathbb{D}$, can form the category End($\mathbb{D}$) which has:

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• e.g. for $\mathbb{D} = \mathcal{V}\text{-}\mathbb{M}\text{at}$, $\text{End}(\mathbb{D}) = \mathcal{V}\text{-Grph}$

• If $\mathbb{D}$ has parallel $\mathcal{I}$-(co)limits, then End($\mathbb{D}$) has them and $\text{End}(\mathbb{D}) \to \mathbb{D}_1$ creates them.

• The forgetful $\text{Mnd}(\mathbb{D}) \to \text{End}(\mathbb{D})$ creates all limits which exist in $\mathbb{D}$.

Free monads

Theorem

Suppose that $\mathbb{D}$ is a fibrant double category with parallel countable coproducts which are preserved by $\odot$ in each variable. Then the forgetful functor $U\colon \mathrm{Mnd}(\mathbb{D}) \to \mathrm{End}(\mathbb{D})$ has a left adjoint.

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Proof.

The forgetful $U\colon \mathrm{Mnd}(\mathbb{D}) \to \mathrm{End}(\mathbb{D})$ constitutes a fibred 1-cell

$$\begin{array}{ccc} \mathrm{Mnd}(\mathbb{D}) & \xrightarrow{\quad U \quad} & \mathrm{End}(\mathbb{D}) \\ & \searrow \quad \swarrow & \\ & \mathbb{D}_0 & \end{array}$$

For every $X \in \mathbb{D}_0$ the restriction $U_X\colon \mathrm{Mnd}(\mathbb{D})_X \to \mathrm{End}(\mathbb{D})_X$ has a left adjoint, because $\mathrm{End}(\mathbb{D})_X = \mathcal{H}(\mathbb{D})(X,X)$ is a monoidal category with $\otimes = \odot$... □

On monadicity of Mnd($\mathbb{D}$)

Adapt the arguments from the case of bicategories, as in the classic:

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The first step is

Proposition

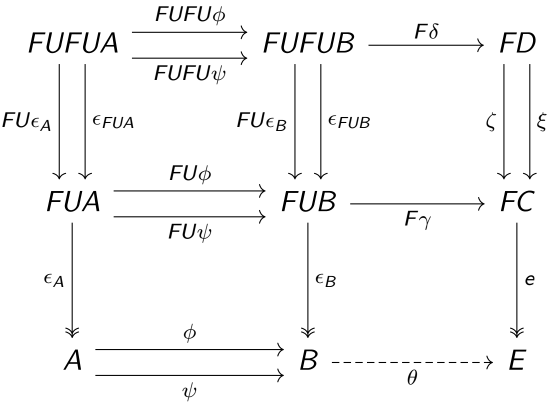
Let $\mathbb{D}$ be a double category which has parallel colimits preserved by $\odot$ in each variable. Then the category Mnd($\mathbb{D}$) has all coequalizers.

Proof.

For a pair of monad morphisms in $\mathbb{D}$ as follows

$$\begin{array}{ccc} X & \overset{A}{\nrightarrow} & X \\ f\downarrow & \Downarrow\phi & \downarrow f \\ Y & \underset{B}{\nrightarrow} & Y \end{array} \qquad \begin{array}{ccc} X & \overset{A}{\nrightarrow} & X \\ g\downarrow & \Downarrow\psi & \downarrow g \\ Y & \underset{B}{\nrightarrow} & Y \end{array}$$

We have the following commutative diagram in $\mathsf{Mnd}(\mathbb{D})$.



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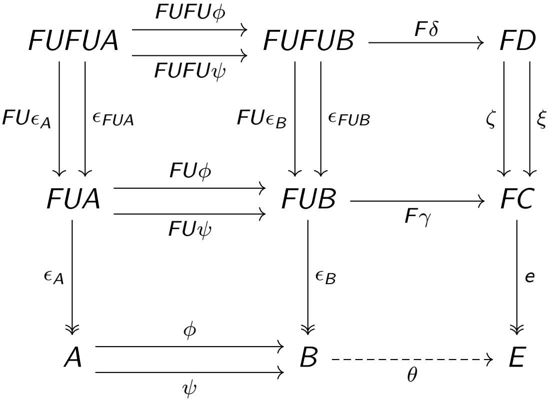

E acquires a monad structure, finish off with 3×3 lemma.

Theorem

Let $\mathbb{D}$ be a double category which has parallel colimits preserved by $\odot$ in each variable. Then the forgetful functor $\mathsf{Mnd}(\mathbb{D}) \to \mathsf{End}(\mathbb{D})$ is monadic.

Proof.

Same 3×3 diagram, but now assume in addition that ϕ, ψ are a U-split. pair. Can apply the 3×3 diagram lemma here to deduce that θ is the coequalizer of $U\phi, U\psi$ in $\mathsf{End}(\mathbb{D})$, i.e. U preserves coequalizers of U-split pairs. □

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So for example we recover,

- $\mathcal{V}$-Cat is monadic over $\mathcal{V}$-Grph, for nice enough $\mathcal{V}$.
- Cat($\mathcal{C}$) is monadic over Grph($\mathcal{C}$), for nice enough $\mathcal{C}$.

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Corollary

Let $\mathbb{D}$ be a double category which has parallel colimits preserved by $\odot$ in each variable. Then the category of monads $\mathsf{Mnd}(\mathbb{D})$ is cocomplete.

Towards local presentability

Can we find general conditions on $\mathbb{D}$ that would ensure $\mathsf{Mnd}(\mathbb{D})$ is locally presentable?

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Can we find general conditions on $\mathbb{D}$ that would ensure $\mathsf{Mnd}(\mathbb{D})$ is locally presentable?

In a different work (V. A-S, C. Vasilakopoulou, *Sweedler Theory for double categories*), we considered a notion of local presentability for a double category.

Roughly, $\mathbb{D}$ is locally λ-presentable if:

- $\mathbb{D}_0$ and $\mathbb{D}_1$ are locally λ-presentable.
- $\mathfrak{s}, \mathfrak{t} \colon \mathbb{D}_1 \to \mathbb{D}_0$ have left and right adjoints.
- $- \odot - \colon \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$ is accessible.

Theorem

Let $\mathbb{D}$ be a locally presentable double category where $\odot$ preserves colimits in each variable. Then the category of monads $\mathsf{Mnd}(\mathbb{D})$ is locally presentable.

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For $\mathbb{D} = \mathcal{V}$-$\mathbb{M}$at this would yield:

Corollary

If $\mathcal{V}$ is a locally presentable monoidal category where $\otimes$ preserves colimits in each variable, then $\mathcal{V}$-Cat is locally presentable.

Thank you for your attention!