Open power-objects in categories of algebras

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$\mathbf{P}1$ is a *subobject classifier*: $\mathbf{Sub}(X) \cong \mathrm{Hom}(X, \mathbf{P}1)$

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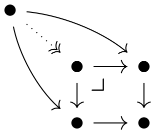

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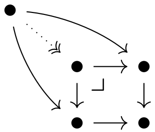

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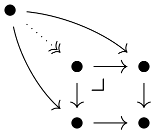

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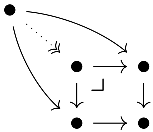

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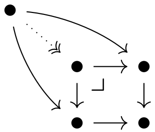

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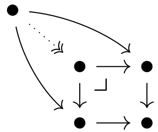

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continuous relation $X \rightsquigarrow Y$:

$$X \xleftarrow{\ \textit{open}\ } \bullet \longrightarrow Y$$

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- how do open power-objects translate in the internal logic?
- how do open power-objects translate in monoidal topology? generalized Vietoris monads? [Hoffman 2014]