The three-dimensional structures formed by monoidal categories, bicategories, double categories, etc.

Nathanael Arkor   James Deikun   Keisuke Hoshino

Category Theory 2025, Brno

Why are two-dimensional categories important?

Just as structured sets form categories, facilitating a uniform study of their properties and the constructions they support, so too do structured categories form two-dimensional categories, facilitating their study.

In other words, once we have accepted that categories are important, two-dimensional categories are inevitable.

Why are three-dimensional categories important?

Similarly, the study of two-dimensional categories (and hence merely of categories) renders the necessity of three-dimensional categories inevitable.

4.2. Remark. Unfortunately, we do not know of any universal property satisfied by this construction. In particular, $\mathbb{H}\text{-}\mathsf{Kl}(\mathbb{X}, T)$ is not a Kleisli object for T in $v\mathcal{D}bl$ in the sense of [Str72a]; the latter would instead contain *vertical* Kleisli arrows. In fact, for general $\mathbb{X}$ there need not even be a canonical functor $\mathbb{X} \longrightarrow \mathbb{H}\text{-}\mathsf{Kl}(\mathbb{X}, T)$.

[CS10]

- [CKSW] talk about a smooth three-dimensional structure of bicategories, lax functors, modules and modulations.
- What is this?

[Par13]

els to pull back along morphisms between theories. For this, we must go beyond the two-dimensional framework developed here to a fully three-dimensional structure encompassing at least double categories, lax functors, lax transformations, modules, and multimodulations. As the proper understanding of categorical logic requires at least a

[LP24]

What structure do categories form?

It is well known that categories, functors, and natural transformations assemble into a 2-category.

However, it is becoming increasingly appreciated that this structure is insufficient for many concepts in category theory, e.g. presheaves, weighted colimits, monadicity, relative adjointness.

The missing puzzle piece is the notion of distributor, which axiomatises heteromorphisms between objects of different categories.

Distributors

A **distributor** $\mathcal{C} \nrightarrow \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ comprises, for each $C \in \mathcal{C}$ and $D \in \mathcal{D}$, a collection of **heteromorphisms**

$$C \rightsquigarrow D$$

together with composition operations with morphisms of $\mathcal{C}$ and $\mathcal{D}$.

This data may be expressed as a functor

$$\mathcal{C}^{op} \times \mathcal{D} \longrightarrow \mathbf{Set}$$

The double category of categories

Small categories, functors ($\to$), distributors ($\nrightarrow$), and natural transformations assemble into a (weak) **double category**.

A 2-cell (left) comprises a natural family (right).

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "P", "\shortmid"{marking}] \arrow[d, "F"'] \arrow[dr, phantom, "\varphi"] & \mathscr{C}' \arrow[d, "G"] \\
\mathscr{D} \arrow[r, "Q"', "\shortmid"{marking}] & \mathscr{D}'
\end{tikzcd}
\qquad\Bigg|\qquad
\begin{tikzcd}
P(c, c') \arrow[d, "\varphi_{c,c'}"] \\
Q(Fc, Gc')
\end{tikzcd}$$

What structure do double categories form?

It is well known that bicategories, pseudofunctors, pseudonatural transformations, and modifications assemble into a tricategory [GPS95].

In other words, weak 2-dimensional globular categories assemble into a weak 3-dimensional globular category.

Analogously, we might hope that weak 2-dimensional cubical categories should assemble into a weak 3-dimensional cubical category.

Distributors between double categories

A distributor between categories expresses a collection of heteromorphisms between objects of two different categories.

Double categories have two kinds of morphism. Consequently, there are two kinds of distributor between double categories, corresponding to two kinds of heteromorphism.

Tight distributors

A tight distributor between double categories $\mathbb{X}$ and $\mathbb{Y}$ comprises

- for each $x \in \mathbb{X}$ and $y \in \mathbb{Y}$, a collection of tight heteromorphisms $x \rightsquigarrow y$
- for each $x \nrightarrow x'$ and $y \nrightarrow y'$, a collection of 2-cells

$$\begin{tikzcd} x \arrow[r, "\shortmid" marking] \arrow[d, squiggly] \arrow[dr, phantom, "\varphi"] & x' \arrow[d, squiggly] \\ y \arrow[r, "\shortmid" marking] & y' \end{tikzcd}$$

(plus composition)

Loose distributors

A loose distributor between double categories $\mathbb{X}$ and $\mathbb{Y}$ comprises

- for each $x \in \mathbb{X}$ and $y \in \mathbb{Y}$, a collection of loose heteromorphisms $x \nrightarrow y$
- for each $x \to x'$ and $y \to y'$, a collection of 2-cells

$$\begin{tikzcd} x \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\varphi"] & y \arrow[d] \\ x' \arrow[r, "\shortmid"{marking}] & y' \end{tikzcd}$$

(plus composition)

An obstruction to composition

We might hope that functors, tight distributors, and loose distributors form the three kinds of 1-cell in a '**weak triple category**' of weak double categories. Unfortunately, this is <u>not</u> the case.

Observation (Shulman)

There does not exist a (weakly) associative composition of distributors between double categories.

Forgoing composition

We cannot compose distributors between double categories. However, this is actually not a problem.

In fact, there exist many structures throughout category theory that exhibit similar phenomena.

- Cospans in categories without pushouts
- Matrices in monoidal categories without sums
- Distributors between non-small categories
- Distributors between enriched categories

The solution

The treatment for lack of composites is to instead describe directly the fundamental structure expressed by composites: multiary morphisms.

This is the same principle at work as when one moves from monoidal categories to multicategories.

What's more, this approach is often desirable even when composites exist, as it leads to a coherence-free description of the structures in question, as well as their lax morphisms.

Virtual double categories

A virtual double category is like a double category, but in which we do not impose the existence of identities or composites of loose morphisms.

Instead, we have 2-cells with multiary domain.

$$\begin{tikzcd}
\bullet \arrow[r, "\shortmid" marking] \arrow[d] & \bullet \arrow[r, "\shortmid" marking] & \bullet \arrow[r, "\shortmid" marking] \arrow[d, phantom, "\varphi"] & \bullet \arrow[r, "\shortmid" marking] & \bullet \arrow[d] \\
\bullet \arrow[rrrr, "\shortmid" marking] & & & & \bullet
\end{tikzcd}$$

Virtual double categories

Just as in a multicategory, we have identities and composites of multiary 2-cells.

$$\begin{tikzcd}
\bullet \arrow[r, "p", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "="] & \bullet \arrow[d, equal] \\
\bullet \arrow[r, "p"', "\shortmid"{marking}] & \bullet
\end{tikzcd}$$

$$\begin{tikzcd}
\bullet \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[drr, phantom, "\varphi_1"] & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[drr, phantom, "\varphi_2"] & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[d] \\
\bullet \arrow[rr, "\shortmid"{marking}] \arrow[d] \arrow[drrrr, phantom, "\psi"] & & \bullet \arrow[rr, "\shortmid"{marking}] & & \bullet \arrow[d] \\
\bullet \arrow[rrrr, "\shortmid"{marking}] & & & & \bullet
\end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd}
\bullet \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[drrrr, phantom, "\psi(\varphi_1, \varphi_2)"] & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[r, "\shortmid"{marking}] & \bullet \arrow[d] \\
\bullet \arrow[rrrr, "\shortmid"{marking}] & & & & \bullet
\end{tikzcd}$$

The virtual double category of categories

Categories, functors, distributors and natural transformations assemble into a virtual double category.

A 2-cell (left) comprises a natural family (right).

$$\begin{tikzcd} \mathcal{C}_0 \arrow[r, "P_1", "\shortmid"{marking}] \arrow[d, "F"'] & \mathcal{C}_1 \arrow[r, "P_2", "\shortmid"{marking}] \arrow[d, phantom, "\varphi"] & \mathcal{C}_2 \arrow[r, "P_3", "\shortmid"{marking}] & \mathcal{C}_3 \arrow[d, "G"] \\ \mathcal{D} \arrow[rrr, "Q"', "\shortmid"{marking}] & & & \mathcal{D}' \end{tikzcd}$$

$$\begin{array}{c} P_1(c_0, c_1) \times P_2(c_1, c_2) \times P_3(c_2, c_3) \\ \Big\downarrow \varphi_{c_0, c_1, c_2} \\ Q(Fc_0, Gc_3) \end{array}$$

The structure that virtual double categories form

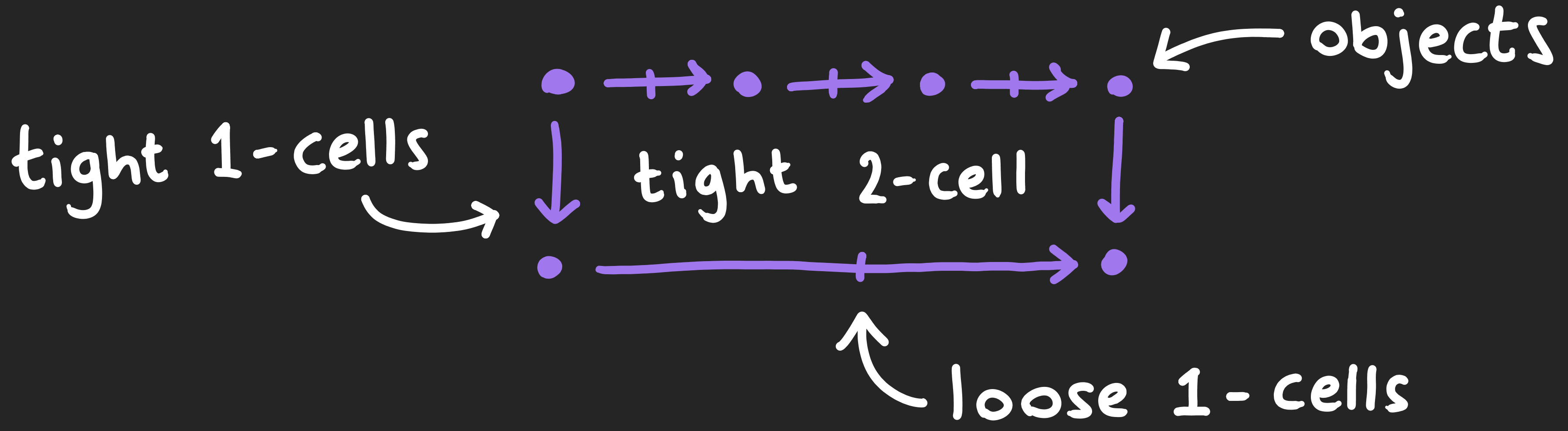
Our insight is that, while weak double categories do not assemble into a weak triple category, virtual double categories do assemble into a virtual triple category, whose 1-cells are the functors, tight distributors, and loose distributors.

But what exactly is a virtual triple category?

Virtual triple categories

A **virtual triple category** comprises:

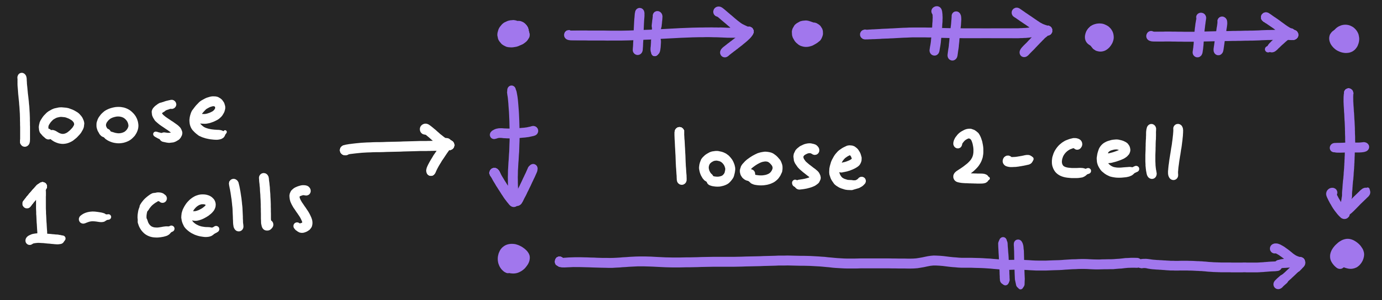
1. A virtual double category



2. A third collection of 1-cells 

Virtual triple categories

3. A collection of 2-cells

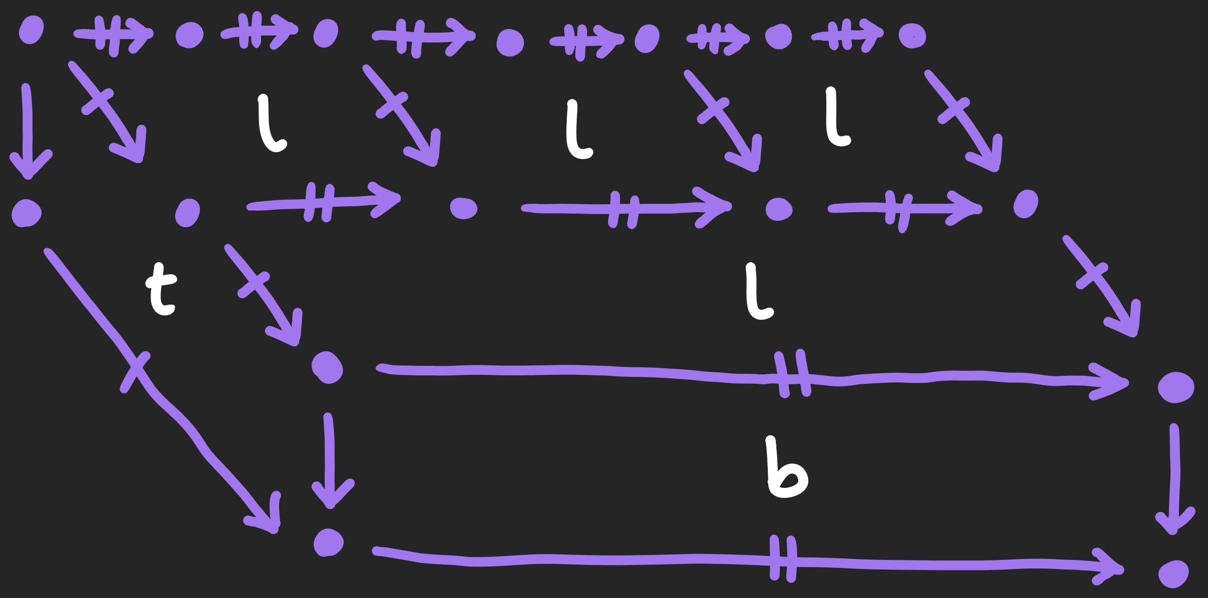


slack 1-cells

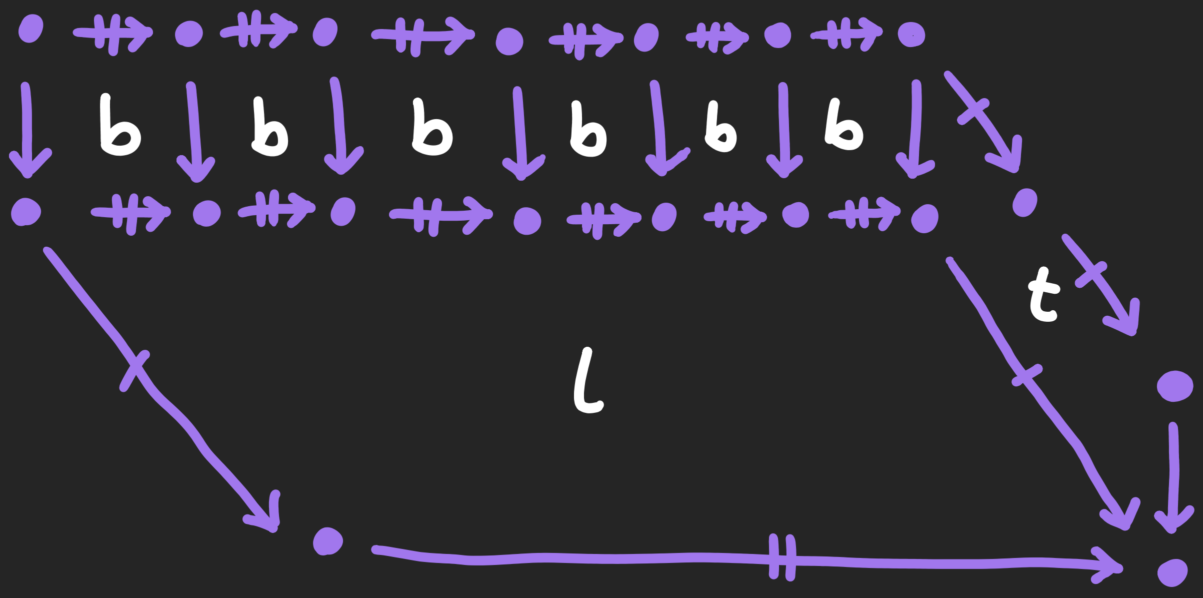
4. A collection of 2-cells

Virtual triple categories

5. A collection of 3-cells



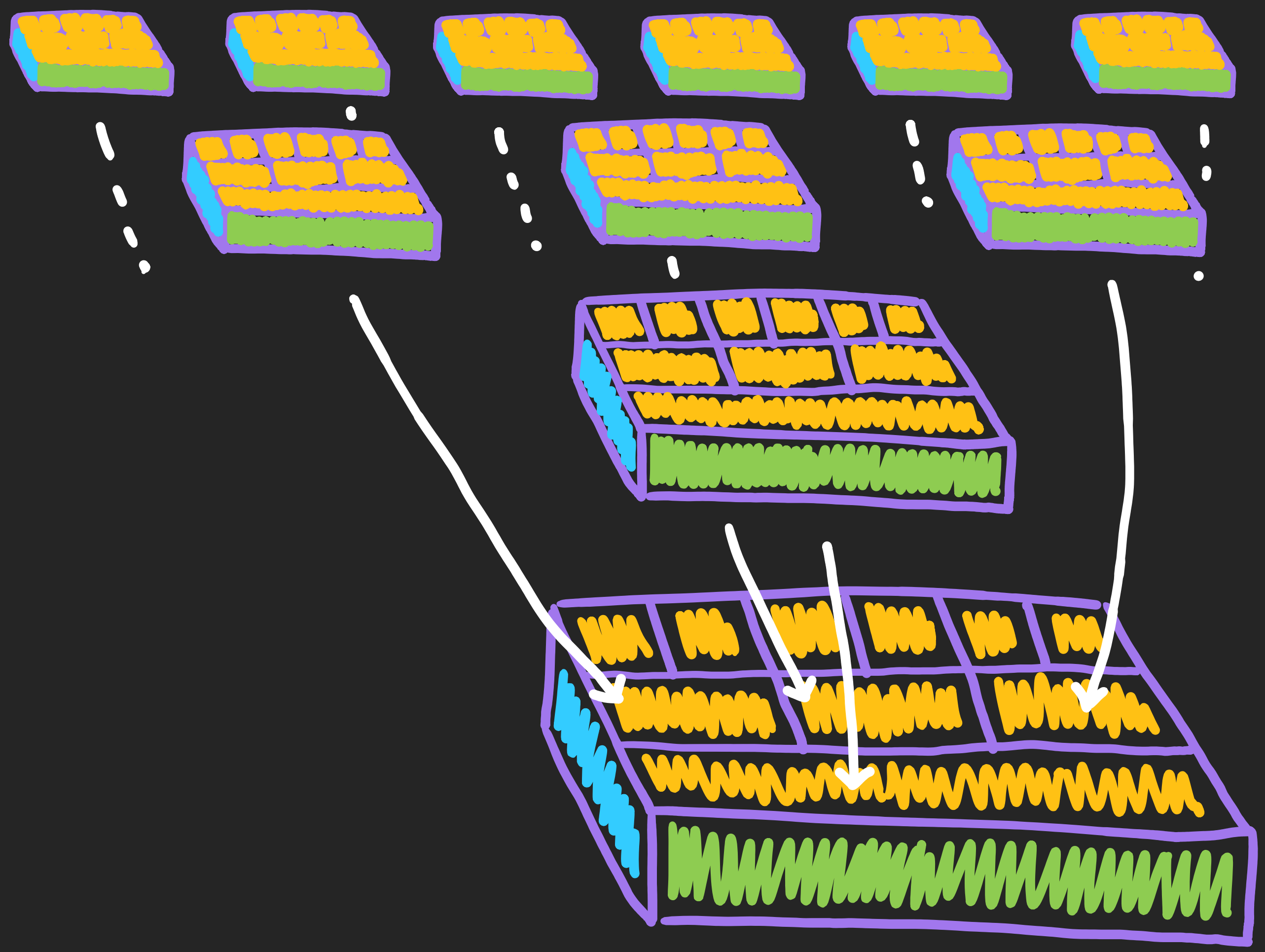
front faces



back faces

Virtual triple categories

6. Composites and identities



Theorem

There is a virtual triple category of virtual double categories.

Some special cases

Before elaborating on our construction, let us take a look at some consequences.

By restricting the virtual triple category of virtual double categories to various subcategories, we obtain rich 3-dimensional structures that both generalise and coherently arrange familiar concepts in ordinary, enriched, and two-dimensional category theory.

The virtual triple category of monoidal categories

There is a virtual triple category whose

- objects are monoidal categories
- tight 1-cells are lax monoidal functors
- loose 1-cells are monoidal distributors
- slack 1-cells are two-sided actions
- tight 2-cells generalise monoidal nat. trans.
- loose 2-cells generalise strong distributors
- boundary 2-cells are strong functors
- 3-cells generalise strong nat. trans.

The virtual triple category of multicategories

There is a virtual triple category whose

- objects are multicategories
- tight 1-cells are functors
- loose 1-cells are multidistributors
- slack 1-cells are two-sided multiactions
- tight 2-cells generalise multinat. trans.
- loose 2-cells generalise strong distributors
- boundary 2-cells are strong functors
- 3-cells generalise strong multinat. trans.

Categories graded on two sides

In 2002, Kelly–Labella–Schmitt–Street introduced two directed generalisations of enriched categories.

- Categories enriched from $\mathcal{V}$ to $\mathcal{W}$ generalise $\mathcal{W}$-enriched categories.
- Categories graded from $\mathcal{V}$ to $\mathcal{W}$ generalise $\mathcal{W}$-graded categories, i.e. $\widehat{\mathcal{W}}$-enriched cat's.

Theorem (Arkor & Lucyshyn-Wright)

Categories graded on two sides $\cong$ tight distributors.

The virtual triple category of bicategories

There is a virtual triple category whose

- objects are bicategories
- tight 1-cells are lax functors
- loose 1-cells are categories graded on two sides
- slack 1-cells are pseudodistributors
- tight 2-cells generalise functors of graded cat's
- loose 2-cells generalise distributors graded o.t.s.
- boundary 2-cells generalise modules
- 3-cells generalise multimodulations

The virtual triple category of double categories

There is a virtual triple category whose

- objects are double categories
- tight 1-cells are lax functors
- loose 1-cells are tight distributors
- slack 1-cells are loose distributors
- tight 2-cells generalise transformations
- loose 2-cells generalise loose transformations
- boundary 2-cells are alterations
- 3-cells generalise multimodulations

Constructing the virtual double category of categories

While it is straightforward to construct the virtual double category of categories by hand, there is a more elegant construction. The key observation is that the composition of a category equips it with the structure of a monad in the virtual double category of sets and spans.

Theorem (Burroni)

The virtual double category of categories and distributors is $\mathbb{M}\mathrm{nd}(\mathbb{S}\mathrm{pan})$.

Constructing the VTC of VDCs

Theorem

Given a virtual triple category $\mathbb{X}$, there is a virtual triple category $\mathbb{M}\mathrm{nd}(\mathbb{X})$ of monads in $\mathbb{X}$.

Definition

The virtual triple category of virtual double categories is $\mathbb{M}\mathrm{nd}(\mathbb{S}\mathrm{pan})$.

Totalities of virtual n-tuple categories

Conjecture

For each $n \leq \omega$, virtual n-tuple categories assemble into a virtual $n+1$-tuple category.

Evidence:

0. Sets assemble into a category.
1. Categories assemble into a virtual double category.
2. This talk.

n. Ongoing work.

Summary

- Weak double categories do not assemble into a weak triple category.
- Virtual double categories do assemble into a virtual triple category.
- As a consequence, we obtain coherent three-dimensional structures of monoidal categories, bicategories, double categories.
- These methods consequently are useful even for globular higher categories.

References

[Bur71] Albert Burroni. 'T-catégories (catégories dans un triple)'. In: *Cahiers de topologie et géométrie différentielle catégoriques* 12.3 (1971), pages 215–321.

[CKSW03] J. R. B. Cockett, J. Koslowski, R. A. G. Seely and R. J. Wood. 'Modules'. In: *Theory and Applications of Categories* 11.17 (2003), pages 375–396.

[CS10] Geoffrey S. H. Cruttwell and Michael A. Shulman. 'A unified framework for generalized multicategories'. In: *Theory and Applications of Categories* 24.21 (2010), pages 580–655.

[GPS95] Robert Gordon, Anthony John Power and Ross Street. *Coherence for tricategories*. Volume 558. American Mathematical Society, 1995.

[KLSS02] Max Gregory Kelly, Anna Labella, Vincent Schmitt and Ross Street. 'Categories enriched on two sides'. In: *Journal of Pure and Applied Algebra* 168.1 (2002), pages 53–98.

[Lei99] Tom Leinster. *Generalized enrichment for categories and multicategories*. 1999. arXiv: math/9901139.

[LP24] Michael Lambert and Evan Patterson. 'Cartesian double theories: A double-categorical framework for categorical doctrines'. In: *Advances in Mathematics* 444 (2024), page 109630.

[Par11] Robert Paré. 'Yoneda theory for double categories'. In: *Theory and Applications of Categories* 25.17 (2011), pages 436–489.

[Par13] Robert Paré. 'The "Triple Category" of Bicategories'. Talk presented at the CMS Summer Meeting. 2013. URL: https://www.mathstat.dal.ca/~pare/Bicat(Beamer).pdf.

The +-construction for generalised multicategories

Let T be a nice monad on a category $\mathcal{E}$. There is a monadic adjunction

$$\begin{tikzcd}
T\text{-}\mathbf{Cat} \arrow[d, shift right] \arrow[d, phantom, "\dashv"] & & & C_1 \arrow[dl] \arrow[dr] & \\
T\text{-}\mathbf{Grph} \arrow[u, shift right] & & TC_0 \arrow[ull, phantom, "\longleftarrow"] & & C_0
\end{tikzcd}$$

inducing a free T-category monad T^+ on T-$\mathbf{Grph}$.

This monad is again nice, permitting the construction to be iterated [Lei99].

Virtual n-tuple categories

$$\begin{tikzcd} \mathrm{Set} \arrow[loop above, "1"] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Cat} \arrow[d, shift right=2] \arrow[d, phantom, "\dashv"] \\ \mathrm{Grph} \arrow[u, shift right=2] \arrow[loop right, "T_2"] \end{tikzcd} \qquad \begin{tikzcd} T_2\text{-}\mathrm{Cat} \arrow[d, shift right=2] \arrow[d, phantom, "\dashv"] \\ T_2\text{-}\mathrm{Grph} \arrow[u, shift right=2] \arrow[loop right, "T_3"] \end{tikzcd} \qquad \begin{tikzcd} T_3\text{-}\mathrm{Cat} \arrow[d, shift right=2] \arrow[d, phantom, "\dashv"] \\ T_3\text{-}\mathrm{Grph} \arrow[u, shift right=2] \arrow[loop right, "T_4"] \end{tikzcd}$$

virtual double categories

categories

(= virtual single categories)

virtual triple categories