A Higher Effective Topos

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joint work with

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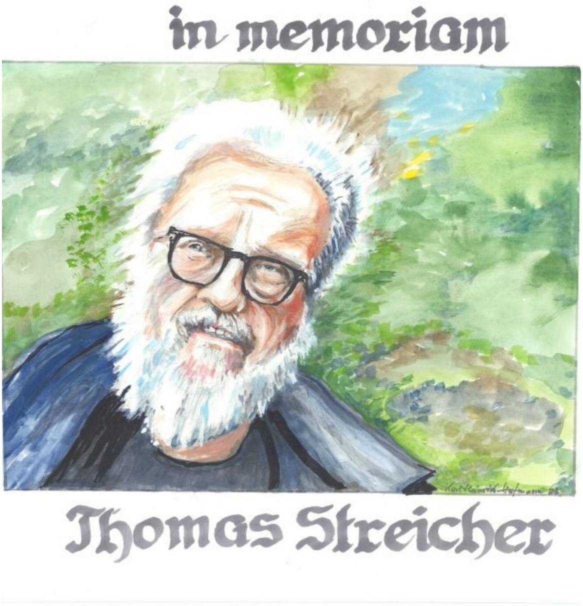

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The effective 1-topos $\mathcal{E}\!f\!f_1$

Recall the effective 1-topos $\mathcal{E}\!f\!f_1$ of Hyland (1982):

1. $\mathcal{E}\!f\!f_1$ is an elementary topos that is not Grothendieck.
2. All maps $f : N \to N$ on the NNO in $\mathcal{E}\!f\!f_1$ are computable.
3. The global sections functor Γ has a fully faithful right adjoint.

$$\Gamma : \mathcal{E}\!f\!f_1 \rightleftarrows \mathsf{Set} : \nabla$$

4. $\mathcal{E}\!f\!f_1$ has a complete internal category $\mathbb{D}$ that is not a poset.
5. Every object E in $\mathcal{E}\!f\!f_1$ is covered by a projective object $P \twoheadrightarrow E$.
6. So $\mathcal{E}\!f\!f_1$ models extensional MLTT with an impredicative universe, quotient types, W-types, and enough projectives.

A higher effective topos $\mathcal{E}\!f\!f_\infty$

We have a higher version $\mathcal{E}\!f\!f_\infty$ such that:

1. $\mathcal{E}\!f\!f_\infty$ is a non-Grothendieck "elementary higher topos" with
$$\mathcal{E}\!f\!f_1 \simeq (\mathcal{E}\!f\!f_\infty)_{\leq 0} \subset \mathcal{E}\!f\!f_\infty\,.$$
2. N in $\mathcal{E}\!f\!f_1$ is still an NNO in $\mathcal{E}\!f\!f_\infty$ for which all maps $f : N \to N$ are computable.
3. The global sections functor Γ factors through *truncated spaces* $\mathcal{S}_{<\infty} \subset \mathcal{S}$ and then has a right adjoint.
$$\Gamma : \mathcal{E}\!f\!f_\infty \;\rightleftarrows\; \mathcal{S}_{<\infty} : \nabla$$
4. There are complete internal categories $\mathbb{D}_1 \subset \mathbb{D}_2 \subset \cdots$, and each $\mathbb{D}_n$ is properly an n-category.
5. Every object E in $\mathcal{E}\!f\!f_\infty$ is covered by a projective $P \twoheadrightarrow E$.
6. $\mathcal{E}\!f\!f_\infty$ models* HoTT with impredicative, univalent universes.

$\mathcal{E}\!f\!f_1$ as an exact completion

The 1-category $\mathcal{E}\!f\!f_1$ is the ex/lex completion of the category $\mathbb{P}$ of *partitioned assemblies*.

$$\mathbb{P} \xhookrightarrow{\ reg/lex\ } \mathsf{Asm} \xhookrightarrow{\ ex/reg\ } \mathcal{E}\!f\!f_1$$

As subcats of the colimit completion $\mathsf{y} : \mathbb{P} \hookrightarrow \widehat{\mathbb{P}}$:

$\mathbb{P}$	indecomposable projectives $\mathsf{y}P$
Asm	image factorizations $\mathsf{y}P \twoheadrightarrow A \rightarrowtail \mathsf{y}Q$
$\mathcal{E}\!f\!f_1$	exact quotients of assemblies $A' \rightrightarrows A \twoheadrightarrow E$

The third step uses a result of Lack (1999).

Two theorems about exact completions

Recall the following:

Theorem (Carboni, 1995)

If $\mathbb{C}$ is weakly LCC, then $\mathbb{C}^{\mathrm{ex/lex}}$ is LCC.

Theorem (Menni, 2000)

If $\mathbb{C}$ has a "generic proof", then $\mathbb{C}^{\mathrm{ex/lex}}$ has a subobject classifier.

Since $\mathbb{P}$ is weakly LCC and has a generic proof, its ex/lex completion $\mathcal{E}\mathit{ff}_1 = \mathbb{P}^{\mathrm{ex/lex}}$ is a topos.

Generalization in two steps

We generalize the 1-exact completion $\mathcal{E}\mathit{ff}_1 = \mathbb{P}^{(\mathrm{ex}/\mathrm{lex})}$ in two steps:

i. $\mathcal{E}\mathit{ff}_2 = \mathbb{P}^{(2\mathrm{ex}/\mathrm{lex})}$ = coherent presheaves of 1-groupoids on $\mathbb{P}$.

ii. $\mathcal{E}\mathit{ff}_\infty = \mathbb{P}^{(\infty\mathrm{ex}/\mathrm{lex})}$ = coherent presheaves of ∞-groupoids on $\mathbb{P}$.

We shall have "elementary higher n-toposes",

$$\mathcal{E}\mathit{ff}_1 \subset \mathcal{E}\mathit{ff}_2 \subset \ldots \subset \mathcal{E}\mathit{ff}_\infty$$

each with $\mathcal{E}\mathit{ff}_n = (\mathcal{E}\mathit{ff}_{n+1})_{<n}$.

So $\mathcal{E}\mathit{ff}_1 = (\mathcal{E}\mathit{ff}_2)_{<1}$ is the 1-category of 0-types in a 2-topos.

A warning about too much exact completion

It might be thought that we could present $\mathcal{E}\mathit{ff}_\infty$ by $\mathcal{E}\mathit{ff}_1^{\Delta^{op}}$ (with the Kan–Quillen model structure), but there is a problem.

The constructive Kan–Quillen model structure (of Gambino et al.) does not give what we want:

Since $\mathcal{E}\mathit{ff}_1$ is already an exact completion, and $\mathcal{E}\mathit{ff}_1^{\Delta^{op}}$ is an ∞-exact completion of that, the subcategory of 0-types in $\mathcal{E}\mathit{ff}_1^{\Delta^{op}}$ will be bigger than $\mathcal{E}\mathit{ff}_1$.

Coherent presheaves

Definition

- A presheaf Q is *quasicompact* if it is covered by a representable
$$\mathrm{y}P \twoheadrightarrow Q$$
- A map $Y \to X$ of presheaves is *quasicompact* if its pullbacks over representables are all quasicompact.
$$\begin{tikzcd} Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\ \mathrm{y}P \arrow[r] & X \end{tikzcd}$$
- A presheaf C is *coherent* if both it and its diagonal
$$C \longrightarrow C \times C$$
are quasicompact.

$\mathcal{E}\!f\!f_1$ as coherent presheaves

Theorem

The presheaves E in $\mathcal{E}\!f\!f_1 \subset \widehat{\mathbb{P}}$ are exactly the coherent ones,

$$\mathcal{E}\!f\!f_1 = \widehat{\mathbb{P}}_{coh} \subset \widehat{\mathbb{P}}.$$

For the proof, recall that in with $\mathbb{P} \hookrightarrow \mathsf{Asm} \subset \mathcal{E}\!f\!f_1 \subset \widehat{\mathbb{P}}$ we had:
E is in $\mathcal{E}\!f\!f_1$ iff for assemblies A, A' there is

$$A' \rightrightarrows A \to E \qquad \text{exact.}$$

The result then follows by unwinding the definitions.

NB: We recover the description of $\mathcal{E}\!f\!f_1 = \mathbb{P}^{(ex/lex)}$ in terms of pseudo-equivalence relations $Q \rightrightarrows P$ in $\mathbb{P}$ by covering both E and the kernel of the cover.

$$\mathrm{y}Q \twoheadrightarrow K \rightrightarrows yP \twoheadrightarrow E$$

$\mathcal{E}\!f\!f_2$ as coherent stacks

This suggests presenting $\mathcal{E}\!f\!f_2$, the 2ex/lex completion[1] of $\mathbb{P}$, by the *coherent groupoids* in $\widehat{\mathbb{P}}$. These are the presheaves of groupoids that are quasicompact with quasicompact diagonals:

Definition

A presheaf of groupoids $G : \mathbb{P}^{op} \to \mathrm{Gpd}$ is *coherent* iff it is pointwise equivalent to a strict one $K = (K_1 \rightrightarrows K_0)$ such that:

1. K_0 is a quasicompact object $\mathrm{y}P \twoheadrightarrow K_0$,
2. the first diagonal $K_1 \to K_0 \times K_0$ is quasicompact,
3. the second diagonal $K_1 \to K_1 \times_{K_0 \times K_0} K_1$ is quasicompact,

$$\begin{tikzcd} K_1 \arrow[r] & K_1 \times_{K_0 \times K_0} K_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & K_1 \arrow[d] \\ & K_1 \arrow[r] & K_0 \times K_0 \end{tikzcd}$$

[1] as a (2,1)-category

$\mathcal{E}\mathit{ff}_2$ as coherent stacks

Theorem (A.–Emmenegger 2025)

The 2-category $\mathcal{E}\mathit{ff}_2$ is presented by the 1-category of coherent presheaves of groupoids,

$$[\mathbb{P}^{\mathsf{op}}, \mathsf{Gpd}]_{coh}$$

with the (restricted) "strong stacks" model structure of Joyal–Tierney (1991). Its 0*-types recover the effective 1-topos.*

$$\mathcal{E}\mathit{ff}_1 = (\mathcal{E}\mathit{ff}_2)_{<1}$$

$\mathcal{E}\!f\!f_\infty$ as coherent ∞-stacks

Similarly, we can describe the ∞ex/lex completion[2] of $\mathbb{P}$ as the full subcategory $\mathcal{E}\!f\!f_\infty \subset [\mathbb{P}^{\mathrm{op}}, \mathcal{S}]$ consisting of coherent presheaves of ∞-groupoids on $\mathbb{P}$:

Theorem (AAB 2025)

$\mathcal{E}\!f\!f_\infty$ is equivalent to the following equal subcategories of $[\mathbb{P}^{\mathrm{op}}, \mathcal{S}]$:

1. *The coherent objects: the E that are truncated, quasicompact, and with all higher diagonals $E \to E^{S^n}$ quasicompact.*
2. *The truncated objects E with all $\pi_n E$ in $\mathcal{E}\!f\!f_1 \subset \mathcal{E}\!f\!f_\infty$.*
3. *The truncated objects that are colimits of Kan complexes in $\mathbb{P}$.*
4. *The closure of $\mathbb{P}$ under quotients of Segal groupoids.*

The proof is similar to some recent works of Anel, Lurie, Stefanich.

[2] as an $(\infty, 1)$-category

A type-theoretic formulation

We also have the following type-theoretic formulation.

1. Let $\mathcal{U}$ in $[\mathbb{P}^{\mathrm{op}}, \mathcal{S}]$ be a (large enough) univalent universe (Shulman 2019).
2. Let $\mathcal{P} \subset \mathcal{U}$ be the subuniverse of representable maps.
3. Define a type $Q : \mathcal{U}$ to be *quasicompact* if there merely exist $P : \mathcal{P}$ and a cover $P \twoheadrightarrow Q$.
4. Define the subuniverses $\mathcal{E}_n \subset \mathcal{U}_{\leq n}$ of *coherent* n-types by induction:
   - E is in $\mathcal{E}_{-2}$ if it is contractible;
   - E is in $\mathcal{E}_{n+1}$ if it is quasicompact and all $x =_E y$ are in $\mathcal{E}_n$.
5. We have $\mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \bigcup_n \mathcal{E}_n =: \mathcal{E}_\infty \subset \mathcal{U}$.

We then take global sections to obtain the n-categories $\mathcal{E}\mathit{ff}_n$ and

$$\mathcal{E}\mathit{ff}_\infty = \mathrm{Hom}(1, \mathcal{E}_\infty).$$

Properties of $\mathcal{E}\mathit{ff}_\infty$

The category $\mathcal{E}\mathit{ff}_\infty$ has the following:

- finite coproducts $X + Y$
- quotients of Segal groupoids $\ldots \substack{\rightarrow\\\rightarrow\\\rightarrow} G_1 \rightrightarrows G_0 \to Q$
- all n-truncations $\|X\|_n$
- exponentials Y^X and dependent products $\Pi_{x:X} Y_x$
- subcategories $\mathit{Eff}_1 \simeq (\mathcal{E}\mathit{ff}_\infty)_{<1}$ and $\mathcal{E}\mathit{ff}_2 \simeq (\mathcal{E}\mathit{ff}_\infty)_{<2}$
- an NNO $1 \to N \to N$ from $\mathcal{E}\mathit{ff}_1$
- a subobject classifier $1 \rightarrowtail \Omega$ from $\mathcal{E}\mathit{ff}_1$
- impredicative univalent universes $\mathbb{D}_0 \subset \mathbb{D}_1 \subset \cdots$
- each $\mathbb{D}_n$ has inductive types $W_{x:X} D_x$

The category $\mathcal{E}\mathit{ff}_\infty$ does *not* have:

- infinite coproducts
- all pushouts (e.g. no S^2)
- any untruncated objects

Properties of $\Gamma \dashv \nabla$

- $\nabla : \mathcal{S}_{<\infty} \to \mathcal{E}\!f\!f_\infty$ is fully faithful and exact.
- $\Gamma : \mathcal{E}\!f\!f_\infty \to \mathcal{S}_{<\infty}$ is the $\neg\neg$-localization.
- Γ has a *partial* left adjoint $\Delta : \mathcal{S}_\pi \to \mathcal{E}\!f\!f_\infty$ on π-finite spaces. It is LCC, exact, and preserves sums.

$$
\begin{tikzcd}
\mathcal{S}_{<\infty} \arrow[r, shift left, "\nabla"] & \mathcal{E}\!f\!f_\infty \arrow[l, shift left, "\Gamma"] \\
\mathcal{S}_\pi \arrow[u, hook] \arrow[ur, "\Delta"'] &
\end{tikzcd}
$$

- The *discrete objects* $D \in \mathbb{D}_n$ are the n-truncated ones with

$$\nabla 2 \perp D\,.$$

- Equivalently, D is discrete if all $\pi_k D$ are in $\mathbb{D}_0$, which consists of the subquotients of N.

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