COGRAPHS, TWOFOLD SYMMETRIC MONOIDAL STRUCTURES, & FACTORIZATION ALGEBRAS

Clark Barwick, The University of Edinburgh

CT2025, Brno – 16 July 2025

Available at https://clarkbar.github.io/papers/facts.pdf

LOCALITY

Locality: measurements made in a region of spacetime should not depend upon measurements performed in a **distant** region of spacetime. (Haag, Haag–Kastler)

More refined: the cluster decomposition principle. (Weinberg)

Factorization algebras (Beilinson–Drinfeld, Francis–Gaitsgory, Costello–Gwilliam, Raskin) allow us to combine observables in spacetime regions that are sufficiently **apart**.

SAME & DIFFERENT

Homotopy theory studies **sameness** as a **structure** rather than a **property**.

Let X be a space and $x, y \in X$. The **path space** $\ulcorner x \sim y \urcorner$ is the space of ways for x and y to be the same. These path spaces are then assembled in combinatorially useful ways.

Let's try to study **difference** or **apartness** in the same way. We'll call it **isolation theory**. We'll define spaces $\ulcorner x \nsim y \urcorner$ of ways for x and y to be **different**. We'll assemble these spaces in combinatorially useful ways.

EXAMPLE

Consider a manifold M, and consider its homotopy type $\Pi(M)$. Let $x, y \in M$. The path space is the fiber of the diagonal

$$\begin{array}{ccc} \ulcorner x \sim y \urcorner & \longrightarrow & \Pi(\Delta_M) \\ \downarrow & & \downarrow \\ \{(x, y)\} & \longrightarrow & \Pi(M \times M) \end{array}$$

Note: if $M = \mathbf{R}^n$, then $\ulcorner x \sim y \urcorner$ is always contractible, even if $x \neq y$.

EXAMPLE – CONTINUED

Let us define this dual thing as the fiber of the **complement** of the diagonal:

$$\begin{array}{ccc} \ulcorner x \nsim y \urcorner & \longrightarrow & \Pi(M \times M - \Delta_M) \\ \downarrow & & \downarrow \\ \{(x, y)\} & \longrightarrow & \Pi(M \times M) \end{array}$$

Note: if $M = \mathbf{R}^n$, then $\ulcorner x \nsim y \urcorner$ is always S^{n-1}, even if $x = y$.

Let's now think about how to assemble these spaces combinatorially.

GRAPHS

Graph (V, E): a pair consisting of

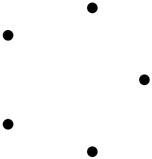
- a finite set V of **vertices**
- a set $E \subseteq \binom{V}{2}$ of **edges**

Morphism $(V, E) \to (V', E')$: a map $f\colon V \to V'$ such that

$$\{a, b\} \in E \implies \{f(a), f(b)\} \in E'$$

NOTATION

$\langle n \rangle$ = the graph with n vertices and no edges



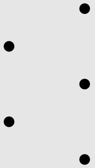
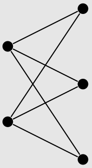
We disallow loops! So there is no morphism from K_2 =  to $\langle 1 \rangle$.

SUMS

If A and B are two graphs, we can add them in two ways:

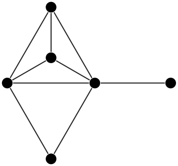
- disconnected sum, or coproduct: $A \underline{\oplus} B$
- connected sum, or join: $A \overline{\oplus} B$

Example

$\langle 2 \underline{\oplus} 3 \rangle =$  $= \langle 5 \rangle$ $\langle 2 \overline{\oplus} 3 \rangle =$  $= K_{2,3}$

FUN

$$\left\langle 1 \,\overline{\oplus}\, \left(1 \,\underline{\oplus}\, \left(1 \,\overline{\oplus}\, \left(1 \,\underline{\oplus}\, \left(1 \,\overline{\oplus}\, 1\right)\right)\right)\right)\right\rangle =$$



TWOFOLD SYMMETRIC MONOIDAL STRUCTURES

The sums $\underline{\oplus}, \overline{\oplus}$ are symmetric monoidal structures with the same unit, $\langle 0 \rangle = \varnothing$.
There is also a natural intertwiner

$$(A \mathbin{\overline{\oplus}} B) \mathbin{\underline{\oplus}} (C \mathbin{\overline{\oplus}} D) \to (A \mathbin{\underline{\oplus}} C) \mathbin{\overline{\oplus}} (B \mathbin{\underline{\oplus}} D)$$

which exhibits

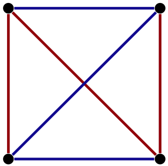
- $\overline{\oplus}$ as lax symmetric monoidal with respect to $\underline{\oplus}$ and
- $\underline{\oplus}$ as colax symmetric monoidal with respect to $\overline{\oplus}$.

In other words, the category of graphs is a **twofold symmetric monoidal category**. (Balteanu–Fiedorowicz–Schwänzl–Vogt)

COGRAPHS

A **cograph** is a graph in which the following implication holds for all vertices a, b, c, d:

$$\{\{a,b\},\{b,c\},\{c,d\}\} \subseteq E \implies \{\{a,c\},\{a,d\},\{b,d\}\} \cap E \neq \varnothing$$



Fun fact: cographs = graphs that can be presented using

$$\langle 1 \rangle, \quad \underline{\oplus}, \quad \overline{\oplus}$$

(**co** is an unfortunate shortening of **complement reducible**.)

UNIVERSAL PROPERTY

D = the twofold symmetric monoidal category of cographs

$\langle 1 \rangle \in \mathrm{CMon}(\mathbf{D}, \underline{\oplus})$

Theorem

$\boldsymbol{D}$ is the free twofold symmetric monoidal category on a commutative monoid for the left tensor product:

$$\mathrm{TwofoldSMF}((\boldsymbol{D}, \underline{\oplus}, \overline{\oplus}), (\boldsymbol{C}, \underline{\otimes}, \overline{\otimes})) = \mathrm{CMon}(\boldsymbol{C}, \underline{\otimes})$$

EXAMPLE

Let's return to our manifold M.

For every cograph $\langle\lambda\rangle$, consider the space

$$M^{\lambda} = \{x \in M^{V\langle\lambda\rangle} : \{a, b\} \in E\langle\lambda\rangle \implies x_a \neq x_b\}$$

The assignment $\langle\lambda\rangle \mapsto M^{\lambda}$ is a functor $\mathbf{D}^{op} \to \mathbf{Man}$.

$$M^n = M \times \cdots \times M$$

$$M^{1 \overline{\oplus} 1} = M \times M - \Delta_M$$

ISOLABILITY STRUCTURES

Consider an inclusion $\langle\lambda\rangle \subseteq \langle\lambda'\rangle$ of a full (‘induced’) subgraph, and let $\langle\lambda\rangle \to \langle\mu\rangle$ be any morphism that is surjective on vertices. Then there’s a pushout

$$\begin{array}{ccc} \langle\lambda\rangle & \hookrightarrow & \langle\lambda'\rangle \\ \big\downarrow & & \big\downarrow \\ \langle\mu\rangle & \hookrightarrow & \langle\mu'\rangle \end{array}$$

in **D**.

An **isolability object** of a category **X** is a functor $\mathbf{D}^{op} \to \mathbf{X}$ such that all pushout squares as above in **D** are carried to pullback squares in **X**.

ISOLABILITY SPACES FROM MANIFOLDS

Let M be a manifold.
The assignment $\langle\lambda\rangle \to M^\lambda$ is an isolability manifold.
Passing to homotopy types gives us an **isolability ∞-groupoid**.
Passing to stratified homotopy types gives us an **isolability ∞-category**.

In the case $M = \mathbf{R}^n$, the corresponding isolability ∞-category admits a combinatorial description: it carries $\langle\lambda\rangle$ to the category of 'n-structures' on $\langle\lambda\rangle$.

ISOLABILITY STRUCTURES ON HILBERT SCHEMES

Let X be a variety over a field.
Let H be the moduli space of closed subvarieties of X (the 'Hilbert algebraic space').
For every cograph $\langle\lambda\rangle$, we let H^{λ} be the moduli space of tuples $(Z_a)_{a\in V\langle\lambda\rangle}$ of closed subvarieties such that if $\{a,b\}\in E\langle\lambda\rangle$, then $Z_a\cap Z_b=\varnothing$.
Now $\langle\lambda\rangle\mapsto H^{\langle\lambda\rangle}$ is an isolability object in algebraic spaces.

GEOMETRY OF ISOLABILITY SPACES

Let's now assume that we are working with a site or topos **X**.
We equip the objects of **X** with a notion of **sheaf**: a lax symmetric monoidal sheaf

$$\mathbf{A}\colon \mathbf{X}^{op} \to \mathbf{Pr}^L$$

Perhaps the simplest example is $\mathbf{X} = \mathbf{An}$, and $\mathbf{A} = Fun(-, \mathbf{An})$.
Other sheaf theories that can be represented this way include: stacks, constructible sheaves, *D*-modules, etc.

We want to extend our sheaf theory to isolability objects.

Applying **A** objectwise to an isolability object $X^{\bullet}\colon \mathbf{D}^{op} \to \mathbf{X}$, we get a lax twofold symmetric monoidal functor

$$\mathbf{A}(X^{\bullet})\colon \mathbf{D} \to \mathbf{Pr}^{L}$$

In concrete terms, we have two external tensor products

$$\overline{\boxtimes}\colon \mathbf{A}(X^{\lambda}) \otimes \mathbf{A}(X^{\mu}) \to \mathbf{A}(X^{\lambda \underline{\oplus} \mu})$$

which is where the symmetric monoidal structure on each individual $\mathbf{A}(X^{\lambda})$ comes from, and

$$\underline{\boxtimes}\colon \mathbf{A}(X^{\lambda}) \otimes \mathbf{A}(X^{\mu}) \to \mathbf{A}(X^{\lambda \overline{\oplus} \mu})$$

which is something **new** that comes from isolation theory.

FACTORIZATION ALGEBRAS – INFORMALLY

Recall that we have a category **X** and a sheaf theory $\mathbf{A}\colon \mathbf{X}^{op} \to \mathbf{Pr}^L$.

Let $X^\bullet\colon \mathbf{D}^{op} \to \mathbf{X}$ be an isolability object. We obtain a lax twofold symmetric monoidal functor $\mathbf{A}(X^\bullet)\colon \mathbf{D} \to \mathbf{Pr}^L$.

Informally, a **factorization algebras on $X^\bullet$ with coefficients in A** now consists of the following:

- for every cograph $\langle\lambda\rangle$, a sheaf $F_\lambda \in \mathbf{A}(X^\lambda)$
- for every surjective map of cographs $\phi\colon \langle\lambda\rangle \to \langle\mu\rangle$, an identification $\phi^{**}F_\lambda = F_\mu$
- for every pair of cographs $\langle\lambda\rangle$ and $\langle\mu\rangle$, an identification

$$F_{\lambda \overline{\oplus} \mu} = F_\lambda \underline{\boxtimes} F_\mu$$

- plus all coherence …

FACTORIZATION ALGEBRAS – FORMALLY

To define this precisely we remember $\mathbf{A}(X^\bullet)$ only as a nonunital lax symmetric monoidal functor $(\mathbf{D}_S, \overline{\oplus}) \to \mathbf{Pr}^L$. Now a factorization algebra is a nonunital lax symmetric monoidal transformation from the constant diagram **An** to $\mathbf{A}(X^\bullet)$.
This recovers existing versions of this notion, and it works in quite a lot of generality.

Two questions.

- How should one deal with **units** as treated by Gaitsgory and Raskin?
- Factorization algebras make sense as a functor from the bicategory of lax twofold symmetric monoidal functors $\mathbf{D} \to \mathbf{Pr}^L$ to the bicategory of categories. Is it **representable**?