Lax Monoidal Structures from Monoidal Structures

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Main Result

Theorem

Given a monoidal category $(\mathcal{D}, \otimes, I)$*, and any full subcategory* $\mathcal{C}$*, then the presheaf category* $\hat{\mathcal{C}} := Set^{\mathcal{C}^{op}}$ *inherits a lax monoidal structure.*

Lax Definition

Lax Monoidal Structure

Given a category $\mathcal{C}$, an (unbiased) lax monoidal structure consists of the data:

1. Arrows $\otimes_m : \mathcal{C}^{\times m} \to \mathcal{C}$ for all $m \in \mathbb{N}$
2. Natural transfromations $\mu_\xi : \otimes_n \circ (\otimes_{m_1} \times ... \times \otimes_{m_n}) \Rightarrow \otimes_m$ for all partitions $\xi : m_1 + ... + m_n = m$
3. A natural transformation $\eta : 1_\mathcal{C} \Rightarrow \otimes_1$

subject to coherence conditions.

If $\eta = id_{1_\mathcal{C}}$ then we call the lax monoidal structure **Strictly Normal**.

Main Theorem

Theorem

Given a Monoidal Category $(\mathcal{D}, \otimes, I)$ then any full subcategory $\mathcal{C}$ inherits a lax promonoidal structure.

Corollary (Day-Street [DS03])

The presheaf category $\hat{\mathcal{C}} := Set^{\mathcal{C}^{op}}$ inherets a lax monoidal structure.

Pro Definition

Profunctor

A profunctor $F : \mathcal{A} \nrightarrow \mathcal{C}$ between categories is a functor of the form $F : \mathcal{C}^{op} \times \mathcal{A} \to Set$.

Composition of profunctors is given by using a coend, as in: Given $F : \mathcal{A} \nrightarrow \mathcal{B}$ and $G : \mathcal{B} \nrightarrow \mathcal{C}$, then $G \circ F : \mathcal{A} \nrightarrow C$ is given by

$$G \circ F(c, a) = \int^{b \in \mathcal{B}} F(a, b) \times G(b, c).$$

(Lax) Promonoidal Structure

Same definition as a (lax) monoidal structure, but replace every instance of a functor for a profunctor.

Constructing the Lax Promonoidal Structure

Given a monoidal category $(\mathcal{D}, \otimes, I)$, we define the lax promonoidal structure on any full subcategory $\mathcal{C}$ as:

- Given a $n \in \mathbb{N}$, we define the profunctor $\otimes_n^{\mathcal{C}} : \mathcal{C}^{\times n} \nrightarrow \mathcal{C}$ as

$$\mathcal{C}^{op} \times \mathcal{C}^{\times n} \xhookrightarrow[i^{op} \times i^n]{} \mathcal{D}^{op} \times \mathcal{D}^{\times n} \xrightarrow[id_{\mathcal{D}}^{op} \times \otimes_{\text{n-fold}}]{} \mathcal{D}^{op} \times \mathcal{D} \xrightarrow[\mathcal{D}(-,-)]{} Set.$$

- The natural transformation μ is given by composition.
- We use Cruttwell-Shulman [CS10] to show that the coherence conditions are satisfied.

Examples

Theorem

Given a monoidal category $(\mathcal{D}, \otimes, I)$, and any full subcategory $\mathcal{C}$, then the presheaf category $\hat{\mathcal{C}} := Set^{\mathcal{C}^{op}}$ inherits a lax monoidal structure.

Preliminary Examples:

1. $(\mathcal{D}, \otimes, I) = (Op, \otimes_{BV}, *)$, $\mathcal{C} = \Omega$, and $\hat{\mathcal{C}} =$ Dendroidal Sets (Moerdijk-Weiss [MW07]).
2. $(\mathcal{D}, \otimes, I) = (2\text{-}Cat, \boxtimes_{gray}, *)$, $\mathcal{C} = \Theta_2$, and $\hat{\mathcal{C}} = \theta_2 -$Sets (Maehara [Mae21]).
3. $(\mathcal{D}, \otimes, I) = (Cat, \times, *)$, $\mathcal{C} = \Delta$, and $\hat{\mathcal{C}} =$ Simplicial Sets.

Future work

- Assuming that the monoidal category $\mathcal{D}$ have a model structure. Give a model structure on $\hat{\mathcal{C}}$ such that the lax monoidal structure is homotopical in the sense of Heuts-Hinich-Moerdijk [HHM16].
- Explore extensions to the enriched setting.

- G.S.H. Cruttwell and M. Shulman, *A unified framework for generalized multicategories*, Theory Appl. Categ. 24 (2010).
- B. Day and R. Street, *Lax monoids, pseudo-operads, and convolution.*, Contemporary Mathematics 318 (2003).
- S. Eilenberg and J. A. Zilber, *Semi-simplicial complexes and singular homology*, Annals of Mathematics 51:3 (1950).
- G. Heuts, V. Hinich, and I. Moerdijk, *On the equivalence between lurie's model and the dendroidal model for infinity-operads.*, Adv. Math., 302 (2016).
- J. Lurie, *Higher algebra*, Preprint (2017), https://www.math.ias.edu/∼lurie.
- Y. Maehara, *The gray tensor product for 2-quasi-categories*, Advances in Mathematics 377 (2021).
- I. Moerdijk and I. Weiss, *Dendroidal sets*, Algebr. Geom. Topol. 7 (2007).