Naturality for higher-dimensional path types

CT 2025, Brno

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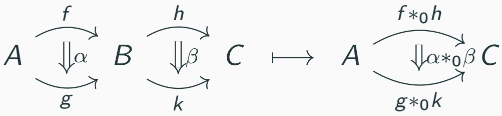
What does functoriality means for weak ω-categories ?

Start with :

- A composition operation in weak ω categories

$$A \xrightarrow{f} B \xrightarrow{g} C \quad \longmapsto \quad A \xrightarrow{f *_0 g} C$$



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$$A \xrightarrow{f} B \xrightarrow{g} C \longmapsto A \xrightarrow{f *_0 g} C \qquad A \underset{g}{\overset{f}{\Downarrow \alpha}} B \underset{k}{\overset{h}{\Downarrow \beta}} C \longmapsto A \underset{g *_0 k}{\overset{f *_0 h}{\Downarrow \alpha *_0 \beta}} C$$

- A set of maximal dimensional arguments of the chosen operation

$$X = \{f\} \qquad X = \{\alpha, \beta\}$$

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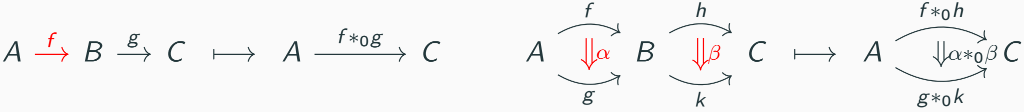

- A set of maximal dimensional arguments of the chosen operation

$$X = \{f\} \qquad\qquad X = \{\alpha, \beta\}$$

Define the functoriality of the chosen operation with respect to the chosen arguments

Functoriality of composite operations

For the composite operation

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \quad \longmapsto \quad A \xrightarrow{(f *_0 g) *_0 h} C$$

pic : 

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$$A \underset{f^+}{\overset{f^-}{\Downarrow \vec{f}}} B \xrightarrow{g} C \xrightarrow{h} D \quad \longmapsto \quad (\vec{f} *_0 g) *_0 h$$

pic :

What about the identity ?

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- Try to apply the same recipe :

$$\color{red}{A^- \xrightarrow{\vec{A}} A^+} \qquad \mapsto \qquad \begin{array}{c} A^- \xrightarrow{\mathrm{id}_{A^-}} A^- \\ \\ A^+ \xrightarrow[\mathrm{id}_{A^+}]{} A^+ \end{array}$$

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Compositions of identities

- Composition of identity 1-cells

$$A \longmapsto A \xrightarrow{\mathrm{id}_A} \xrightarrow{\mathrm{id}_A} A$$

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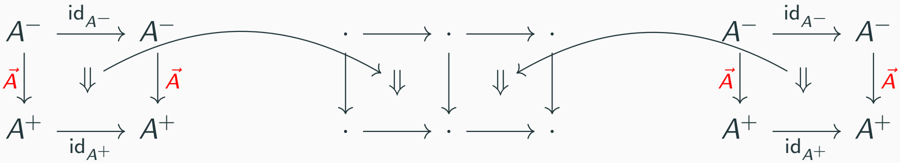

Compositions of cylinders

- We have just illustrated that naturality produces a filler

$$
\begin{array}{ccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
{\scriptstyle h}\downarrow & \Downarrow\alpha & \downarrow{\scriptstyle k} & \Downarrow\beta & \downarrow{\scriptstyle l} \\
E & \xrightarrow{p} & F & \xrightarrow{q} & G
\end{array}
\quad\mapsto\quad
\begin{array}{ccc}
A & \xrightarrow{c(f,g)} & C \\
{\scriptstyle h}\downarrow & \Downarrow s & \downarrow{\scriptstyle l} \\
E & \xrightarrow[c(p,q)]{} & G
\end{array}
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A & \xrightarrow{f} & B & \xrightarrow{g} & C & & A & \xrightarrow{c(f,g)} & C \\
h\big\downarrow & \Downarrow_{\alpha} & \big\downarrow k & \Downarrow_{\beta} & \big\downarrow l & \mapsto & h\big\downarrow & \Downarrow_{s} & \big\downarrow l \\
E & \xrightarrow[p]{} & F & \xrightarrow[q]{} & G & & E & \xrightarrow[c(p,q)]{} & G
\end{array}
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- Treat this as a blackbox and choose $X = \{f, p, \alpha, g, q, \beta\}$

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- We get a filler for the cylindrical composition :