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Coalgebraic models of a Lawvere theory

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Hopf algebras

Bialgebras

How can Lawvere theories help in studying them?

Hopf braces

Moufang bialgebras

Outline

1 Linearizing process

2 General properties

3 Ω-Hopf algebras

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$\mathcal{F}$-coalgebras

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Definition

A coalgebra over a field $\mathcal{K}$ is a triple (C, Δ, ϵ) where

- C is a $\mathcal{K}$-vector space,
- $\Delta\colon C \to C \otimes C$ is a coassociative comultiplication,
- $\epsilon\colon C \to \mathcal{K}$ is a counit.

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Let $\mathcal{F}$ be a type of algebras, namely a set of operation symbols with an arity.

Definition

An $\mathcal{F}$-coalgebra is a coalgebra (C, Δ, ϵ) in Coalg with a homomorphism of coalgebras $f^C \colon C^{\otimes n} \to C$ for any operation f of arity n in $\mathcal{F}$.

Linearized terms

Linearized terms

If C is an $\mathcal{F}$-coalgebra and $T(X)$ the term $\mathcal{F}$-algebra on n variables, we can define a homomorphism

$$\begin{aligned} l_n \colon T(X) &\longrightarrow \mathrm{Hom}(C^{\otimes n}, C) \\ x_i &\longmapsto \epsilon \otimes \cdots \otimes \mathrm{id} \otimes \cdots \otimes \epsilon. \end{aligned}$$

If s and t are two terms in $T(X)$, we call the maps $l_n(s)$ and $l_n(t)$ linearized terms and we say that an $\mathcal{F}$-coalgebra satisfies the equation $s \approx t$ if and only if $l_n(s) = l_n(t)$.

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Example

Let $\mathcal{F} = \{\cdot, 1, S\}$ be the type of groups and $(C, \Delta, \epsilon, \cdot^C, 1^C, S^C)$ an $\mathcal{F}$-coalgebra. Take the term $x \cdot S(x)$, the corresponding linearized term $l_n(x \cdot S(x))$ is

$$\begin{aligned} \cdot^C \circ (\mathrm{id} \otimes S^C) \circ \Delta \colon C &\longrightarrow C \\ a &\longmapsto a_1 \otimes S(a_2). \end{aligned}$$

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Theorem (Perez-Izquierdo, 2007)

Let Σ be a set of equations and $s \approx t$ a consequence of Σ. If an $\mathcal{F}$-coalgebra C satisfies all identities in Σ, then $l_n(s) = l_n(t)$, i.e. it satisfies $s \approx t$.

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We denote by $[\mathcal{T}, \mathsf{Coalg}]$ the category of coalgebraic models of $\mathcal{T}$.

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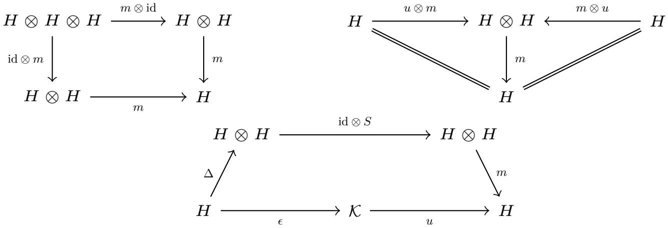
A coalgebraic model for $\mathcal{T}_{\mathsf{Grp}}$ is a (cocommutative) coalgebra (H, Δ, ϵ) with coalgebra maps $m \colon H \otimes H \to H$, $u \colon K \to H$, $S \colon H \to H$ satisfying:



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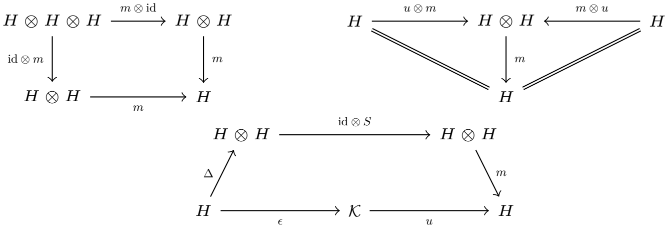

Coalgebraic models of the theory of groups are cocommutative Hopf algebras:

$$[\mathcal{T}_{\mathsf{Grp}}, \mathsf{Coalg}] \cong \mathsf{Hopf}_{\mathsf{coc}}.$$

Hopf braces

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A skew brace (Vendramin, 2016) is a triple $(A, +, \cdot)$ where $(A, +)$ and $(A, \cdot)$ are groups such that the following compatibility condition:

$$a \cdot (b + c) = a \cdot b - a + a \cdot c$$

holds for all a, b, c in A. We denote by SKB the category of skew braces by $\mathcal{T}_{\mathsf{SKB}}$ its corresponding Lawvere theory.

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Coalgebraic models of the theory of skew braces are cocommutative Hopf braces:

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Free functor

Let $\mathcal{T}$ and $\mathcal{S}$ be two Lawvere theories and $G\colon \mathcal{S} \to \mathcal{T}$.
G induces a functor

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Examples

- $\mathcal{S} = \mathsf{FinSet}$, $\mathcal{T} = \mathcal{T}_{\mathsf{Grp}}$ (Takeuchi,1971)

$$\mathsf{Hopf}_{\mathsf{coc}} \underset{F}{\overset{U}{\underset{\leftarrow}{\overset{\rightarrow}{\top}}}} \mathsf{Coalg}$$

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- $\mathcal{S} = \mathcal{T}_{\mathsf{Grp}}$, $\mathcal{T} = \mathcal{T}_{\mathsf{SKB}}$ (Agore and Chirvasitu, 2025)

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A coideal of A is a linear subspace I s.t. $\Delta(I) \subset I \otimes A + A \otimes I$ and $\epsilon(I) = 0$.
I is a $\mathcal{T}$-ideal if it is an ideal with respect to the operations in $\mathcal{T}$.
The $\mathcal{T}$-ideal generated by I exists and we denote it by $\langle I \rangle_{\mathcal{T}}$.

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Coequalizers

Let $A \overset{g}{\underset{f}{\rightrightarrows}} B$ be two maps in $[\mathcal{T}, \mathsf{Coalg}]$. The coequalizer of f and g is the linear quotient $q \colon B \to {}^{B}\!/_{\langle I \rangle_{\mathcal{T}}}$ where $I := \{f(a) - g(a) \mid a \in A\}$.

Full functors

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Proposition

The free functor $[\mathcal{T}, \mathsf{Coalg}] \to [\mathcal{S}, \mathsf{Coalg}]$ can be described as a coequalizer in $[\mathcal{S}, \mathsf{Coalg}]$ and realizes $[\mathcal{T}, \mathsf{Coalg}]$ as a Birkhoff subcategory of $[\mathcal{S}, \mathsf{Coalg}]$.

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Achtung

The Birkhoff's variety theorem fails in the coalgebraic context.

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There is an algebraic functor

$$\mathsf{HBr}_{\mathsf{coc}} \longrightarrow \mathsf{Hopf}_{\mathsf{coc}}.$$

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Ω-Hopf algebras

We can study pointed categories of coalgebraic models, with an algebraic functor to $\mathsf{Hopf}_{\mathsf{coc}}$.

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How to characterize the corresponding theories?

Definition (Higgins, 1956)

An Ω-group is a group $(G, \cdot, u)$ with a set of operations I (with positive arity) and any $f \in I$ satisfies the axiom $f(u, \ldots, u) = u$.

Ω-Hopf algebras

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Coalgebraic models of a theory of Ω-groups

An Ω-Hopf algebra is a Hopf algebra $(H, \cdot, u, \Delta, \epsilon, S)$ endowed with a set of maps F satisfying the following properties:

- an element $t \in F$ is a map $t \colon H^{\otimes n} \to H$ with n positive integer,
- every map $t \colon H^{\otimes n} \to H$ in F is a homomorphism of coalgebras,
- for every n-ary operation $t \in F$ the axiom $t \circ \Delta^{(n)} \circ u = u$ holds.

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Proposition

A category of coalgebraic models of $\mathcal{T}$ is a variety of Ω-Hopf algebras if and only if it is pointed and there exists an algebraic functor $[\mathcal{T}, \mathsf{Coalg}] \to \mathsf{Hopf}_{\mathsf{coc}}$.

Newman's correspondence

Let Hker be the kernel in the category $\mathsf{Hopf}_{\mathsf{coc}}$ and ker the usual kernel of vector spaces.

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Factorization in $\mathsf{Hopf}_{\mathsf{coc}}$

The category $\mathsf{Hopf}_{\mathsf{coc}}$ has a regular epi-mono factorization and the image of $f\colon A \to B$ is given by

$$\frac{A}{A(\mathrm{Hker}\, f)^{+}} = \frac{A}{\ker f} \cong f[A].$$

The equality $A(\mathrm{Hker}\, f)^{+} = \ker f$ is a consequence of the Newman's correspondence between sub-Hopf algebras and $\mathcal{T}_{\mathsf{Grp}}$-ideals.

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Proposition

Let $[\mathcal{T}, \mathsf{Coalg}]$ be variety of Ω-Hopf algebras. Any morphism in $[\mathcal{T}, \mathsf{Coalg}]$ factors through a regular epimorphism followed by a monomorphism.

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Proof.

Take a map $f\colon A \to B$ in $[\mathcal{T}, \mathsf{Coalg}]$. As in $\mathsf{Hopf}_{\mathsf{coc}}$, we can factor f through the cokernel of its kernel:

$$A \xrightarrow{\ f\ } B, \qquad A \xrightarrow{\ e\ } \frac{A}{\langle \mathrm{Hker}\, f^{+}\rangle_{\mathcal{T}}} \xrightarrow{\ m\ } B$$

Let us prove that $\langle \mathrm{Hker}\, f^{+}\rangle_{\mathcal{T}} = \ker f$. Since $\ker f$ is a $\mathcal{T}$-ideal and $\ker f \supseteq \mathrm{Hker}\, f^{+}$, we get the inclusion $\langle \mathrm{Hker}\, f^{+}\rangle_{\mathcal{T}} \subseteq \ker f$. On the other hand, it is clear that $\langle \mathrm{Hker}\, f^{+}\rangle_{\mathcal{T}} \supseteq A\,\mathrm{Hker}\, f^{+} = \ker f$. Hence, e is trivially a regular epi and the map m is a monomorphism. □

Semi-abelianess of Ω-Hopf algebras

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Proposition

Any variety $[\mathcal{T}, \mathsf{Coalg}]$ of Ω-Hopf algebras is regular.

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We have already proved the existence of a regular epi-mono factorization. Regular epimorphisms are pullback-stable because the forgetful functor preserves limits and preserves and reflects regular epimorphisms. □

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Theorem

Any variety $[\mathcal{T}, \mathsf{Coalg}]$ of Ω-Hopf algebras is semi-abelian.

New semi-abelian categories

DiGrp

A digroup is a triple $(A, +, \cdot)$ where $(A, +)$ and $(A, \cdot)$ are groups sharing the same neutral element.

RadRng

A radical ring is a skew brace $(A, +, \cdot)$ satisfying:

$$\begin{aligned}(a+b)\cdot c =& a\cdot c - c + b\cdot c\\ a+b =& b+a.\end{aligned}$$

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A Hopf digroup is $(H, \cdot, \bullet, 1, \Delta, \epsilon, S, T)$ where $(H, \cdot, 1, \Delta, \epsilon, S)$ and $(H, \bullet, 1, \Delta, \epsilon, T)$ are (cocommutative) Hopf algebras.

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All the involved categories are semi-abelian and all the involved arrows are inclusions of Birkhoff subcategories.

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