Lax-idempotent monads in homotopy theory

Joint with Fernando Abellán (work in progress)

Thomas Blom

Max Planck Institute for Mathematics, Bonn

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Example: Cocompletions of categories

- Let $\mathcal{C}$ be a locally small $(\infty, 1)$-category and $\mathcal{P}(\mathcal{C})$ its free cocompletion.
- Then $\mathcal{C}$ is cocomplete if and only if $よ\colon \mathcal{C} \to \mathcal{P}(\mathcal{C})$ admits a left adjoint L.

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- A functor $F\colon \mathcal{C} \to \mathcal{D}$ preserves colimits if and only if for any $X\colon I \to \mathcal{C}$, the assembly map

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Example: Cocompletions of categories

- Let $\mathcal{C}$ be a locally small $(\infty, 1)$-category and $\mathcal{P}(\mathcal{C})$ its free cocompletion.
- Then $\mathcal{C}$ is **cocomplete** if and only if $\text{よ}\colon \mathcal{C} \to \mathcal{P}(\mathcal{C})$ admits a left adjoint L.
- A functor $F\colon \mathcal{C} \to \mathcal{D}$ **preserves colimits** if and only if for any $X\colon I \to \mathcal{C}$, the assembly map

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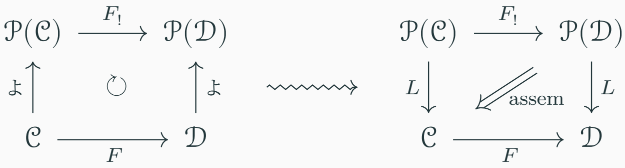
  is an equivalence.
- This is a **Beck–Chevalley condition**:



Example: Cocompletions of categories

This characterizes the sub-$(\infty, 2)$-category $\mathcal{C}\mathrm{at}_{(\infty,1)}^{\mathrm{cocomp}} \subseteq \mathcal{C}\mathrm{at}_{(\infty,1)}$:

- Its objects are the $\mathcal{C}$ for which $よ\colon \mathcal{C} \to \mathcal{P}(\mathcal{C})$ admits a left adjoint.

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- Its **morphisms** are the maps $\mathcal{C} \to \mathcal{D}$ for which the Beck–Chevalley transformation

$$\begin{array}{ccc} \mathcal{P}(\mathcal{C}) & \xrightarrow{F_!} & \mathcal{P}(\mathcal{D}) \\ {\scriptstyle L}\downarrow & \swarrow & \downarrow{\scriptstyle L} \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{D} \end{array}$$

is invertible.

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- A functor $p\colon \mathcal{C} \to \mathcal{B}$ between $(\infty,1)$-categories is a cocartesian fibration if and only if the inclusion

$$\begin{array}{ccc} i\colon \mathcal{C} & \hookrightarrow & \mathcal{C} \times_{\mathcal{B}} \mathrm{Ar}(\mathcal{B}) \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \mathrm{ev}_1} & \\ & \mathcal{B} & \end{array}$$

admits a left adjoint

$$\begin{aligned} \mathrm{tr}\colon \mathcal{C} \times_{\mathcal{B}} \mathrm{Ar}(\mathcal{B}) &\longrightarrow \mathcal{C} \\ (c, f) &\longmapsto f_! c \end{aligned}$$

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is invertible.
- This completely characterizes $\mathcal{C}\mathrm{ocart}(\mathcal{B}) \subseteq \mathcal{C}\mathrm{at}_{(\infty,1)}/\mathcal{B}$.

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    - Their **objects** are those c for which $i\colon c \to Tc$ admits a left adjoint.
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🙌 These are examples of $(\infty, 2)$-categories of algebras over a **lax-idempotent monad**.

Definition (after Kock[1] and Zöberlein[2])

Let $\mathcal{X}$ be an $(\infty,2)$-category and $T\colon \mathcal{X} \to \mathcal{X}$ a monad (i.e. an E_1-algebra in $\mathrm{End}\,\mathcal{X}$).

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Remark

Reversing the 1- or 2-morphisms gives three more variations:

- colax-idempotent monads: $\eta_{Tx} \dashv \mu_x$.
- lax-idempotent comonads: $\varepsilon_{Tx} \dashv \delta_x$.
- colax-idempotent comonads: $\delta_x \dashv \varepsilon_{Tx}$.

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4. The map $U \colon \mathrm{Alg}(T) \to \mathcal{Y}$ is inverse to the canonical map $\mathcal{Y} \to \mathrm{Alg}(T)$. □

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  - For example, it follows that colimits in $\mathcal{C}\mathrm{ocart}(\mathcal{B})$ are computed in $\mathcal{C}\mathrm{at}_{(\infty,1)}/\mathcal{B}$.
- One also obtains adjoint functor theorems for $\mathfrak{X}$ (cf. ADL[3]).

[3] N. Arkor, I. Di Liberti, and F. Loregian. **"Adjoint functor theorems for lax-idempotent pseudomonads".** *Theory and Applications of Categories* 41.20 (2024), pp. 667–685.

Example: Cocompletions revisited

- Since $\mathcal{P}(\mathcal{C})$ is cocomplete, the Yoneda embedding $よ\colon \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{P}(\mathcal{C}))$ always admits a left adjoint $L\colon \mathcal{P}(\mathcal{P}(\mathcal{C})) \to \mathcal{P}(\mathcal{C})$.

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- This defines the **multiplication map** of a lax-idempotent monad structure on $\mathcal{P}\colon \mathcal{C}\mathrm{at}_{(\infty,1)} \to \mathcal{C}\mathrm{at}_{(\infty,1)}$, with $\mathrm{Alg}(\mathcal{P}) = \mathcal{C}\mathrm{at}^{\mathrm{cocomp}}_{(\infty,1)}$.

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- This defines the multiplication map of a lax-idempotent monad structure on $\mathcal{P}\colon \mathcal{C}\mathrm{at}_{(\infty,1)} \to \mathcal{C}\mathrm{at}_{(\infty,1)}$, with $\mathrm{Alg}(\mathcal{P}) = \mathcal{C}\mathrm{at}_{(\infty,1)}^{\mathrm{cocomp}}$.
- This generalizes easily to the following two ways:
  - Given a collection of small $(\infty,1)$-categories $\mathcal{K}$, one can consider the free cocompletion $\mathcal{P}_{\mathcal{K}}(\mathcal{C})$ of $\mathcal{C}$ with respect to $\mathcal{K}$-indexed colimits.
  - One can consider enriched or internal cocompletions.

Example: Operads

- Given a symmetric monoidal $(\infty, 1)$-category $\mathcal{C}$, its **envelope** $\mathrm{Env}(\mathcal{C})$ is defined by the universal property

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- Hence $\mathrm{Env} \colon \mathcal{S}\mathrm{ym}\mathcal{M}\mathrm{on}^{\mathrm{strong}}_{(\infty,1)} \to \mathcal{S}\mathrm{ym}\mathcal{M}\mathrm{on}^{\mathrm{strong}}_{(\infty,1)}$ is a colax-idempotent comonad.

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- Its coalgebras are $(\infty, 1)$-operads:

$$\mathrm{coAlg}(\mathrm{Env}) \simeq \mathcal{O}\mathrm{p}_{(\infty,1)}.$$

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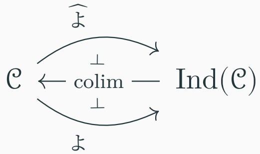
- The corresponding comonad Ind on $\mathcal{C}\mathrm{at}^{\mathrm{fil}}_{(\infty,1)}$ is colax-idempotent.
- $\mathcal{C}$ is a coalgebra for this comonad if $\mathrm{colim}\colon \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ admits a further left adjoint:



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- If such a $\mathcal{C}$ is furthermore accessible, then it is called compactly assembled.
- Efimov[4] showed that the usual K-theory functor extends uniquely to a localizing invariant $K^{\mathrm{cts}}\colon \mathrm{CompAss}_{\mathrm{Stable}} \to \mathrm{Sp}$.

[4] A. I. Efimov. ***K-theory and localizing invariants of large categories.*** 2025. arXiv: `2405.12169`.

What determines a lax-idempotent monad?

- To any lax idempotent monad T on an $(\infty, 2)$-category $\mathfrak{X}$, one can associate the collection of **unit maps**

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$$\mathcal{U} := \{\eta_x \colon x \to Tx\}.$$

- From this collection one can recover the (locally full) $(\infty, 2)$-category $\mathrm{Alg}(T) \subset \mathcal{X}$:
  - Its **objects** are those x for which η_x admits a left adjoint α_x.
  - Its **1-morphisms** are those $f \colon x \to y$ for which $\alpha_y \circ \mathrm{Lan}_{\eta_x}(\eta_y f) \simeq f \circ \alpha_x$.

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- This suggests a 1-1 correspondence between certain collections of arrows in $\mathcal{X}$ and lax-idempotent monads on $\mathcal{X}$.

A non-coherent characterization of lax-idempotent monads

Theorem (after Marmolejo–Wood[5])

Let $\mathcal{U} = \{x \multimap\!\!\!\rightarrow y\} \subseteq \mathrm{Ar}(\mathcal{X})^{\simeq}$ be a collection of arrows such that:

1. $\mathrm{ev}_0 \colon \mathcal{U} \to \mathcal{X}$ is essentially surjective.

[5] F. Marmolejo and R. J. Wood. **"Kan extensions and lax idempotent pseudomonads".** *Theory and*

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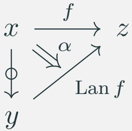
2. 

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3. $\mathrm{Lan}\, f$ is invertible if f lies in $\mathcal{U}$.

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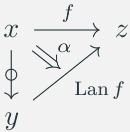
2. 

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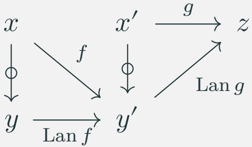
4. 

exhibits $(\mathrm{Lan}\, g) \circ (\mathrm{Lan}\, f)$ as a left Kan extension of $(\mathrm{Lan}\, g) \circ f$.

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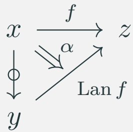
2. 

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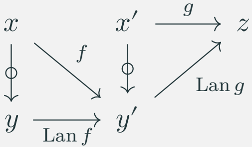
4. 

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Then there exists a lax-idempotent monad on $\mathcal{X}$ given by "$x \mapsto \mathrm{target}(x \multimap\!\!\!\rightarrow y)$", and any lax-idempotent monad arises on $\mathcal{X}$ this way.

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A non-coherent characterization of lax-idempotent monads

Proof sketch.

Let $j\colon x \to\!\!\!\!\circ\!\!\to y$ in $\mathcal{U}$ and $z \in \mathrm{ev}_1(\mathcal{U})$ be given.

Call $y \to z$ extended if it is left Kan extended along $j\colon x \to\!\!\!\!\circ\!\!\to y$. (This turns out to depend only on y and not on j.)

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We define $\mathcal{Y} \subseteq \mathcal{X}$ to be the locally full subcategory with

- **Objects:** $\{z \mid \text{there exists } x \twoheadrightarrow y \text{ in } \mathcal{U} \text{ with } y \simeq z\}$.
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The corresponding monad is **lax-idempotent**. □

🎂 Thank you for listening! 🎂