Weights for Oplax Colimits

Jason Brown
Topos Institute

Presenting work from my thesis, supervised by **Richard Garner**

Defn. (Cat-*weighted colimit*): For a Cat-presheaf $W : \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$ and 2-functor $F : \mathcal{A} \to \mathcal{B}$, the W-*weighted (strict) colimit of* F is a representation:

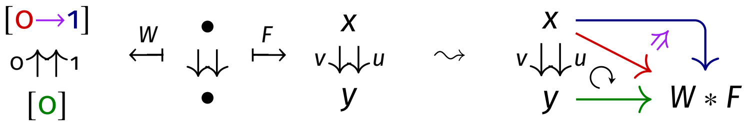
$$\mathcal{B}(W * F, -) \cong \left[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}\right](W, \mathcal{B}(F, -))$$

Defn. (*Oplax colimit*): The W-weighted *oplax* colimit of F is a representation:

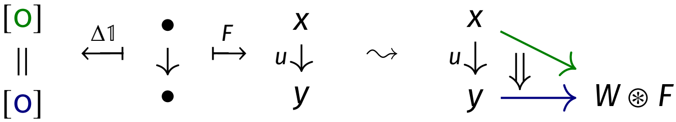
$$\mathcal{B}(W \circledast F, -) \cong \left[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}\right]_{\mathsf{oplax}}(W, \mathcal{B}(F, -))$$

When $W = \Delta\mathbb{1} : \mathcal{A}^{\mathsf{op}} \to \mathsf{Cat}$ we say the the (oplax or otherwise) colimit is *conical*.

Coinserter (strict colimit)



Cograph (conical oplax colimit)



Cographs (conical oplax colimit)

$$\begin{array}{c}[0]\\ \| \\ [0]\end{array} \quad \xleftarrow{\Delta\mathbb{1}} \quad \begin{array}{c}\bullet\\ \downarrow \\ \bullet\end{array} \quad \xmapsto{F} \quad \begin{array}{c}x\\ {\scriptstyle u}\downarrow \\ y\end{array} \qquad \rightsquigarrow \qquad \begin{array}{ccc} x & \searrow & \\ {\scriptstyle u}\downarrow & \Downarrow & \\ y & \longrightarrow & W \circledast F\end{array}$$

are also strict colimits

$$\begin{array}{c}[0\rightarrow 1]\\ \uparrow{\scriptstyle 1} \\ [0]\end{array} \quad \xleftarrow{W} \quad \begin{array}{c}\bullet\\ \downarrow \\ \bullet\end{array} \quad \xmapsto{F} \quad \begin{array}{c}x\\ {\scriptstyle u}\downarrow \\ y\end{array} \qquad \rightsquigarrow \qquad \begin{array}{ccc} x & \searrow & \\ {\scriptstyle u}\downarrow & \Downarrow & \\ y & \longrightarrow & W * F\end{array}$$

Two questions

For a given weight $W : \mathcal{A}^{\mathrm{op}} \to \mathsf{Cat}$, when are:

(a) W-weighted oplax colimits also strict colimits?

(b) W-weighted strict colimits also oplax colimits?

Two questions

For a given weight $W : \mathcal{A}^{\mathrm{op}} \to \mathbf{Cat}$, when are:

(a) W-weighted oplax colimits also strict colimits?

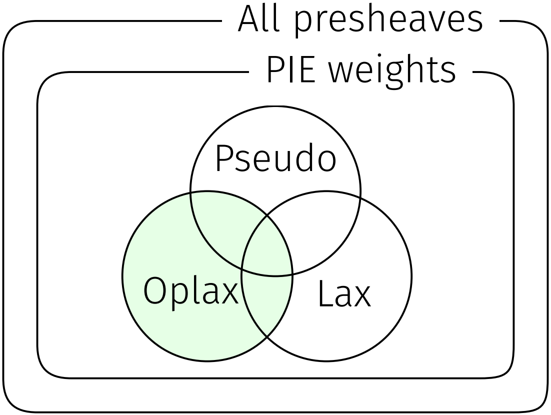
(b) W-weighted strict colimits also oplax colimits?

Answers

(a) Always: $W \circledast F \cong W^{\sharp} * F$ where $W^{\sharp}$ is the *oplax-transformation classifier* for W.

(b) Sometimes: we will call such W "oplax weights".

Oplax weights are presheaves $W : \mathcal{A}^{\mathrm{op}} \to \mathbf{Cat}$ such oplax-colimit-complete 2-categories have W-colimits and oplax-cocontinuous 2-functors preserve W-colimits.



Objectives:

(a) Observe connections to PIE weights

(b) Characterise oplax weights as coalgebras for the *oplax-transformation classifier*, $\sharp$

(c) Relate oplax weights to a comprehensive factorisation system on **2Cat**

(d) Discuss *saturation* properties for related weights

(e) See that $\sharp$-coalgebras with oplax transformations form the free oplax colimit completion of a 2-category

(f) Characterise oplax weights by their categories of elements

There is an adjunction:

$$[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}] \underset{\text{forget}}{\overset{\sharp}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]_{\mathsf{oplax}}$$

We call $\sharp$ the **oplax-transformation classifier**.

There is an adjunction:

$$[\mathcal{A}^{\mathrm{op}}, \mathsf{Cat}] \underset{\text{forget}}{\overset{\sharp}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} [\mathcal{A}^{\mathrm{op}}, \mathsf{Cat}]_{\mathrm{oplax}}$$

We call $\sharp$ the **oplax-transformation classifier**.

For a given $W : \mathcal{A}^{\mathrm{op}} \to \mathsf{Cat}$:

$$\begin{aligned} \mathcal{B}(W \circledast F, -) &\cong \left[\mathcal{A}^{\mathrm{op}}, \mathsf{Cat}\right]_{\mathrm{oplax}}(W, \mathcal{B}(F, -)) \\ &\cong \left[\mathcal{A}^{\mathrm{op}}, \mathsf{Cat}\right]\left(W^{\sharp}, \mathcal{B}(F, -)\right) \cong \mathcal{B}\left(W^{\sharp} * F, -\right) \end{aligned}$$

So $W \circledast F \cong W^{\sharp} * F$.

Any weight of the form $W^{\sharp}$ is an oplax weight.

Note: there is also an adjunction:

$$[\mathcal{A}^{\mathrm{op}}, \mathsf{Cat}] \underset{\text{forget}}{\overset{\natural}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} [\mathcal{A}^{\mathrm{op}}, \mathsf{Cat}]_{\text{pseudo}}$$

We call $\natural$ the **pseudo-transformation classifier** (both as a left adjoint and a comonad.

Weights of the form $W^{\natural}$, i.e. free $\natural$-coalgebras are "pseudo-weights".

The class of *all* $\natural$-coalgebras is the class of *PIE weights*.

PIE weights $W : \mathcal{A}^{\mathsf{op}} \to \mathbf{Cat}$ are equivalently:

(a) coalgebras for the comonad $\natural$ on $[\mathcal{A}^{\mathsf{op}}, \mathbf{Cat}]$

(b) weights in the *saturation* of weights for coproducts coinserters and coequifiers

(c) weights such that the category of elements of $W_0 : \mathcal{A}_0{}^{\mathsf{op}} \to \mathbf{Set}$ has terminal objects in each connected component.

$(b) \Leftrightarrow (c)$: (Power and Robinson 1991)

$(b, c) \Leftrightarrow (a)$: (Lack and Shulman 2012)

We will show that **oplax weights** $W : \mathcal{A}^{op} \to \mathsf{Cat}$ are equivalently:

(a) coalgebras for the comonad $\sharp$ on $[\mathcal{A}^{op}, \mathsf{Cat}]$

(b) weights in the *saturation* of weights for conical oplax colimits

(c) weights such that the category of elements of $W_o : \mathcal{A}_o{}^{op} \to \mathsf{Set}$ has terminal objects in each connected component PLUS some other conditions…

Weights $A^{\mathrm{op}} \longrightarrow \mathbf{Set}$ are equivalent to discrete fibrations on A.

Classes of weights can be characterised by the properties of their discrete fibrations (e.g. PIE weights).

We will use an equivalence between **Cat**-weights and *discrete 2-fibrations* to understand oplax weights.

Defn. (*2-category of elements,* $\mathbf{el}(W)$): for $W\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Cat}$, the 2-category $\mathbf{el}(W)$ has:

0-cells: pairs $(a \in \mathcal{A}, x \in Wa)$

1-cells: $(a, x) \to (b, y)$ are pairs $(u\colon a \to b, f\colon x \to W_u y)$

2-cells: $(u, f) \Rightarrow (v, g)\colon (a, x) \to (b, y)$ are 2-cells $\sigma\colon u \Rightarrow v$ in $\mathcal{A}$ such that $W_\sigma y\, f = g$:

$$\begin{array}{ccc} & \overset{f}{\nearrow} & W_u y \\ x & \circlearrowleft & \downarrow{\scriptstyle W_\sigma y} \\ & \underset{g}{\searrow} & W_v y \end{array}$$

A 2-functor $|W|\colon \mathbf{el}(W) \to \mathcal{A}$ is then given by projection onto the first component, e.g. $|W|(a, x) = a$.

A **discrete 2-fibration** is a split 2-fibration which is a discrete *opfibration* on hom-categories.

There is an equivalence (Lambert 2024):

$$\mathsf{el} : \left[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}\right] \xrightarrow{\simeq} \mathsf{D2Fib}(\mathcal{A})$$

Note: morphisms in $\mathsf{D2Fib}(\mathcal{A})$ are *split-cartesian* functors.

A **discrete 2-fibration** is a split 2-fibration which is a discrete *opfibration* on hom-categories.

There is an equivalence (Lambert 2024):

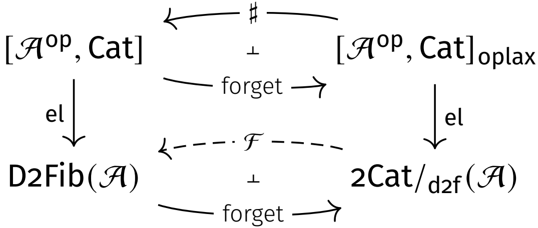
$$\mathsf{el} : \left[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}\right] \xrightarrow{\simeq} \mathsf{D2Fib}(\mathcal{A})$$

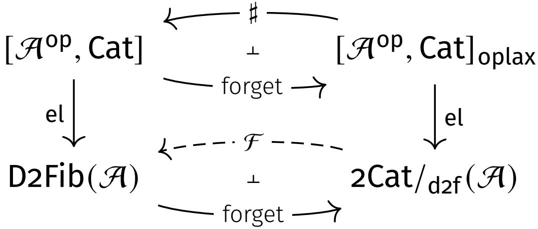
Note: morphisms in $\mathsf{D2Fib}(\mathcal{A})$ are *split-cartesian* functors.

There is *also* an equivalence:

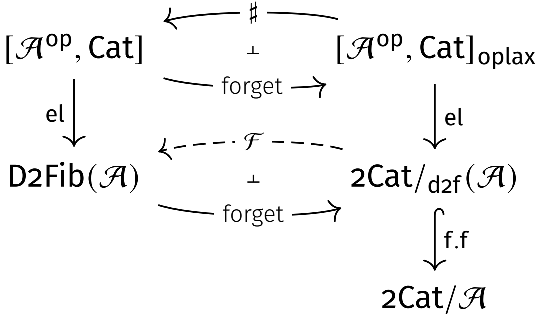
$$\mathsf{el} : \left[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}\right]_{\mathsf{oplax}} \xrightarrow{\simeq} \mathsf{2Cat}/_{\mathsf{d2f}}(\mathcal{A})$$

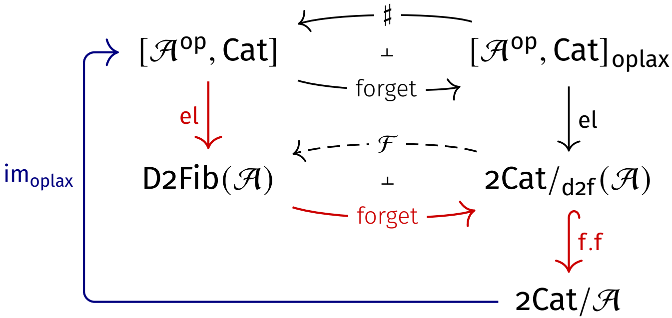
The morphism in $\mathsf{2Cat}/_{\mathsf{d2f}}(\mathcal{A})$ are *all* maps in $\mathsf{2Cat}/\mathcal{A}$.


$[\mathcal{A}^{op}, \mathrm{Cat}]$
$[\mathcal{A}^{op}, \mathrm{Cat}]_{oplax}$
$\sharp$
$\perp$
forget
el
el
$\mathcal{F}$
$\mathrm{D2Fib}(\mathcal{A})$
$\perp$
$\mathrm{2Cat}/_{d2f}(\mathcal{A})$
forget



So $\mathsf{el} : [\mathcal{A}^{\mathrm{op}}, \mathsf{Cat}] \to \mathsf{2Cat}/_{\mathsf{d2f}}(\mathcal{A})$ underlies a coKleisli adjunction for $\sharp$.


[$\mathcal{A}^{op}$, Cat]
[$\mathcal{A}^{op}$, Cat]$_{oplax}$
♯
⊥
forget
el
el
$\mathcal{F}$
D2Fib($\mathcal{A}$)
2Cat/$_{d2f}$($\mathcal{A}$)
⊥
forget
f.f
2Cat/$\mathcal{A}$



el : $[\mathcal{A}^{op}, \mathsf{Cat}] \to \mathsf{2Cat}/\mathcal{A}$ underlies an adjunction for $\sharp$.

Aside: comparison with the 1-dimensional situation:

$$[A^{op}, \mathsf{Set}] \underset{\mathsf{el}}{\overset{\mathsf{im}}{\rightleftarrows}} \mathsf{Cat}/A \qquad [\mathcal{A}^{op}, \mathsf{Cat}] \underset{\mathsf{el}}{\overset{\mathsf{im}_{\mathsf{oplax}}}{\rightleftarrows}} 2\mathsf{Cat}/\mathcal{A}$$

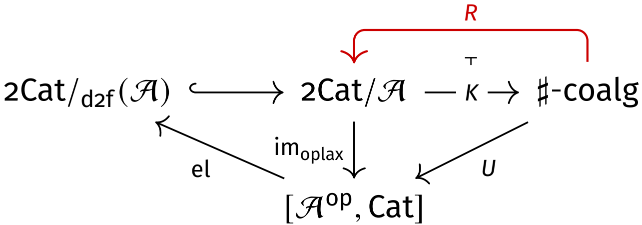
trivial comonad | the comonad $\sharp$

$$\mathsf{im}(F) = \mathsf{colim}\left(B \xrightarrow{F} A \xrightarrow{よ} [A^{op}, \mathsf{Set}]\right)$$

$$\mathsf{im}_{\mathsf{oplax}}(F) = \mathsf{colim}_{\mathsf{oplax}}\left(\mathcal{B} \xrightarrow{F} \mathcal{A} \xrightarrow{よ} [\mathcal{A}^{op}, \mathsf{Cat}]\right)$$

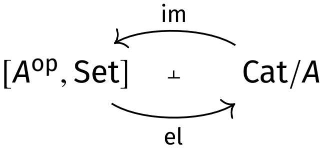
We now have **three** adjunctions for the comonad $\sharp$:

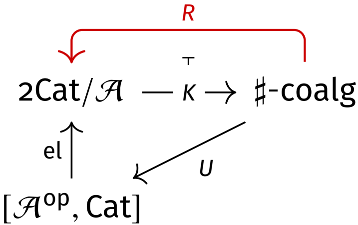
$$\begin{array}{ccccc} \mathsf{2Cat}/_{\mathsf{d2f}}(\mathcal{A}) & \hookrightarrow & \mathsf{2Cat}/\mathcal{A} & \xrightarrow{K} & \sharp\text{-}\mathsf{coalg} \\ & \nwarrow^{\mathsf{el}} & \downarrow{\scriptstyle \mathsf{im}_{\mathsf{oplax}}} & \swarrow_{U} & \\ & & [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}] & & \end{array}$$



Thm. $R : \sharp\text{-}\mathbf{coalg} \to 2\mathbf{Cat}/\mathcal{A}$ is equivalent to the reflective subcategory of 2-functors $\mathcal{B} \to \mathcal{A}$ which are discrete opfibrations on hom-categories (**local discrete opfibrations**).

Aside: orthogonal factorisation systems

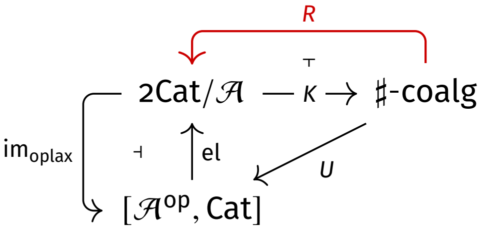




Comprehensive FS:
covering: discrete opfib.s
connected: initial functors
(Street and Walters 1973)

A **2Cat** OFS:
covering: *local* discrete opfib.s
connected: b.o.o locally initial

Classifying ♯-coalgebras



Thm. A presheaf in $[\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}]$ admits a ♯-coalgebra structure precisely if:

(a) it is the oplax image presheaf of a 2-functor $F : \mathcal{B} \to \mathcal{A}$

(b) it is the oplax image presheaf of a local discrete opfibration $F : \mathcal{B} \to \mathcal{A}$

Why do we care about $\sharp$-coalgebras?

Why do we care about $\sharp$-coalgebras?

All $\sharp$-coalgebras are oplax weights

Why do we care about $\sharp$-coalgebras?

All $\sharp$-coalgebras are oplax weights

If W is a $\sharp$-coalgebra, $W \cong \mathsf{im}_{\mathsf{oplax}}(G)$ for some $G : \mathcal{B} \to \mathcal{A}$.

Why do we care about $\sharp$-coalgebras?

All $\sharp$-coalgebras are oplax weights

If W is a $\sharp$-coalgebra, $W \cong \mathrm{im}_{\mathrm{oplax}}(G)$ for some $G : \mathcal{B} \to \mathcal{A}$.

Then for any $F : \mathcal{B} \to \mathcal{C}$:

$$\begin{aligned} W * F &\cong \mathrm{im}_{\mathrm{oplax}}(G) * F \\ &\cong (\mathbb{1} \circledast \mathcal{A}(-, G)) * F \\ &\cong \mathbb{1} \circledast (\mathcal{A}(-, G) * F) \cong \ \mathbb{1} \circledast (F\, G) \end{aligned}$$

Why do we care about $\sharp$-coalgebras?

All $\sharp$-coalgebras are oplax weights

If W is a $\sharp$-coalgebra, $W \cong \mathrm{im}_{\mathrm{oplax}}(G)$ for some $G : \mathcal{B} \to \mathcal{A}$.

Then for any $F : \mathcal{B} \to \mathcal{C}$:

$$\begin{aligned} W * F &\cong \mathrm{im}_{\mathrm{oplax}}(G) * F \\ &\cong (\mathbb{1} \circledast \mathcal{A}(-, G)) * F \\ &\cong \mathbb{1} \circledast (\mathcal{A}(-, G) * F) \cong \ \mathbb{1} \circledast (F\, G) \end{aligned}$$

Note: this isn't true for $\natural$-coalgebras. Not all PIE weights are pseudo weights.

Why do we care about ♯-coalgebras?

All ♯-coalgebras are oplax weights

If W is a ♯-coalgebra, $W \cong \mathrm{im}_{\mathrm{oplax}}(G)$ for some $G : \mathcal{B} \to \mathcal{A}$.

Then for any $F : \mathcal{B} \to \mathcal{C}$:

$$\begin{aligned} W * F &\cong \mathrm{im}_{\mathrm{oplax}}(G) * F \\ &\cong (\mathbb{1} \circledast \mathcal{A}(-, G)) * F \\ &\cong \mathbb{1} \circledast (\mathcal{A}(-, G) * F) \cong \ \mathbb{1} \circledast (F\,G) \end{aligned}$$

Note: this isn't true for ♮-coalgebras. Not all PIE weights are pseudo weights.

If we can show ♯-coalgebras are *saturated*, then *all* oplax weights are ♯-coalgebras.

Aside: all oplax colimits are conical

$$[A^{\mathrm{op}}, \mathsf{Set}] \underset{\mathsf{el}}{\overset{\mathsf{im}}{\underset{\bot}{\leftrightarrows}}} \mathsf{Cat}/A \qquad [\mathcal{A}^{\mathrm{op}}, \mathsf{Cat}] \underset{\mathsf{el}}{\overset{\mathsf{im}_{\mathsf{oplax}}}{\underset{\bot}{\leftrightarrows}}} 2\mathsf{Cat}/\mathcal{A}$$

trivial comonad

the comonad $\sharp$

$$\begin{aligned} W * F &\cong \mathsf{im}(|W|) * F \\ &\cong (\mathbb{1} * A(-, |W|)) * F \\ &\cong \mathbb{1} * (A(-, |W|) * F) \\ &\cong \mathbb{1} * (F\,|W|) \end{aligned}$$

Aside: all oplax colimits are conical

$$[A^{op}, \mathsf{Set}] \underset{\mathsf{el}}{\overset{\mathsf{im}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathsf{Cat}/A \qquad [\mathcal{A}^{op}, \mathsf{Cat}] \underset{\mathsf{el}}{\overset{\mathsf{im}_{\mathsf{oplax}}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} 2\mathsf{Cat}/\mathcal{A}$$

trivial comonad — the comonad $\sharp$

$$\begin{aligned} W * F &\cong \mathsf{im}(|W|) * F \\ &\cong (\mathbb{1} * A(-, |W|)) * F \\ &\cong \mathbb{1} * (A(-, |W|) * F) \\ &\cong \mathbb{1} * (F\,|W|) \end{aligned}$$

$$\begin{aligned} W \circledast F &\cong W^{\sharp} * F \\ &\cong \mathsf{im}_{\mathsf{oplax}}(|W|) * F \\ &\cong (\mathbb{1} \circledast \mathcal{A}(-, |W|)) * F \\ &\cong \mathbb{1} \circledast (\mathcal{A}(-, |W|) * F) \\ &\cong \mathbb{1} \circledast (F\,|W|) \end{aligned}$$

*noted in, for example, (Street 1976)

Defn. The *saturation* Φ^* of a class of weights Φ contains all (small) weights $W : \mathcal{A}^{\mathsf{op}} \to \mathbf{Cat}$ such that Φ-complete/continuous $\implies$ W-complete/continuous.

Φ is *saturated* if $\Phi = \Phi^*$.

Defn. The *saturation* Φ^* of a class of weights Φ contains all (small) weights $W : \mathcal{A}^{\mathsf{op}} \to \mathbf{Cat}$ such that Φ-complete/continuous $\implies$ W-complete/continuous.

Φ is *saturated* if $\Phi = \Phi^*$.

Examples:

$\{\text{non-empty finite coprods}\} \subseteq \{\text{binary coprods}\}^*$

$\{\text{representables}\} = \varnothing^*$

$$\{\text{all small weights}\} = \begin{Bmatrix} \text{coproducts, coequalisers,} \\ \text{tensors by } \mathbb{2} \end{Bmatrix}^*$$

$$\{\text{PIE weights}\} = \begin{Bmatrix} \text{coproducts, coinserters,} \\ \text{coequifiers} \end{Bmatrix}^*$$

Defn. The *saturation* Φ^* of a class of weights Φ contains all (small) weights $W : \mathcal{A}^{\mathsf{op}} \to \mathbf{Cat}$ such that Φ-complete/continuous $\implies$ W-complete/continuous.

Φ is *saturated* if $\Phi = \Phi^*$.

Examples:

$\{\text{non-empty finite coprods}\} \subseteq \{\text{binary coprods}\}^*$

$\{\text{representables}\} = \varnothing^*$

$$\{\text{all small weights}\} = \left\{\begin{matrix}\text{coproducts, coequalisers,} \\ \text{tensors by } \mathbb{2}\end{matrix}\right\}^*$$

$$\{\text{PIE weights}\} = \left\{\begin{matrix}\text{coproducts, coinserters,} \\ \text{coequifiers}\end{matrix}\right\}^*$$

$\{\text{Oplax weights}\} := \left\{W^\sharp\text{'s}\right\}^*$

Consider the following classes of weights:

$$\delta = \left\{\Delta\mathbb{1}^\sharp \mid \mathcal{A}^{\mathsf{op}} \xrightarrow{\Delta\mathbb{1}} \mathsf{Cat},\ \mathcal{A} \in 2\mathsf{Cat}\right\}$$

$$\theta = \left\{W^\sharp \mid \mathcal{A}^{\mathsf{op}} \xrightarrow{W} \mathsf{Cat}, \mathcal{A} \in 2\mathsf{Cat}\right\}$$

$$\Theta = \{\sharp_{\mathcal{A}}\text{-coalgebras} \mid \mathcal{A} \in 2\mathsf{Cat}\}$$

Consider the following classes of weights:

$$\delta = \left\{\Delta\mathbb{1}^\sharp \mid \mathcal{A}^{\mathsf{op}} \xrightarrow{\Delta\mathbb{1}} \mathsf{Cat},\ \mathcal{A} \in 2\mathsf{Cat}\right\}$$

$$\theta = \left\{W^\sharp \mid \mathcal{A}^{\mathsf{op}} \xrightarrow{W} \mathsf{Cat}, \mathcal{A} \in 2\mathsf{Cat}\right\}$$

$$\Theta = \{\sharp_{\mathcal{A}}\text{-coalgebras} \mid \mathcal{A} \in 2\mathsf{Cat}\}$$

(a) $\delta \subset \theta \subset \Theta$, so $\delta^* \subset \theta^* \subset \Theta^*$

(b) $\Theta \subseteq \delta^*$, so $\Theta^* \subseteq (\delta^*)^* = \delta^*$

Consider the following classes of weights:

$$\delta = \left\{ \Delta\mathbb{1}^\sharp \mid \mathcal{A}^{\mathsf{op}} \xrightarrow{\Delta\mathbb{1}} \mathsf{Cat},\ \mathcal{A} \in 2\mathsf{Cat} \right\}$$

$$\theta = \left\{ W^\sharp \mid \mathcal{A}^{\mathsf{op}} \xrightarrow{W} \mathsf{Cat}, \mathcal{A} \in 2\mathsf{Cat} \right\}$$

$$\Theta = \{\sharp_{\mathcal{A}}\text{-coalgebras} \mid \mathcal{A} \in 2\mathsf{Cat}\}$$

(a) $\delta \subset \theta \subset \Theta$, so $\delta^* \subset \theta^* \subset \Theta^*$

(b) $\Theta \subseteq \delta^*$, so $\Theta^* \subseteq (\delta^*)^* = \delta^*$

(c) $\delta^* = \theta^* = \Theta^* =$ oplax weights

Thm: The class Θ of $\sharp$-coalgebras is saturated.

Proof idea.

(a) $\mathsf{U}\colon \sharp\text{-}\mathbf{coalg}_{\mathbf{oplax}} \to [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$ is f.f. ($\sharp$ is oplax idem.)

(b) U creates δ-colimits (Thm. 4.8, Lack 2005)

(c) U is equivalent to $\Theta_{\mathcal{A}} \subseteq [\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$

(d) So $\Theta_{\mathcal{A}}$ is "closed under δ-colimits in $[\mathcal{A}^{\mathsf{op}}, \mathsf{Cat}]$"

(e) So $\Theta = \delta^* = \Theta^*$

Thm: The class Θ of $\sharp$-coalgebras is saturated.

Corollary: $\delta^* = \theta^* = \Theta$.

Corollary: δ and θ are *pre-saturated* (i.e. weights in δ^*/θ^* are δ/θ^*-colimits of representables).

Corollary: $\mathcal{A} \hookrightarrow \Theta_{\mathcal{A}} \simeq \sharp$-**coalg**$_{\mathbf{oplax}}$ is the free cocompletion of $\mathcal{A}$ under oplax colimits.

Objectives:

(a) Observe connections to PIE weights

(b) Characterise oplax weights as coalgebras for the *oplax-transformation classifier*, $\sharp$

(c) Relate oplax weights to a comprehensive factorisation system on **2Cat**

(d) Discuss *saturation* properties for related weights

(e) See that $\sharp$-coalgebras with oplax morphisms form the free oplax colimit completion of a 2-category

(a) Characterise oplax weights by their categories of elements

When is a weight $W : \mathscr{A}^{\mathsf{op}} \to \mathbf{Cat}$ oplax?

Recall: A weight W is *PIE* precisely when $\mathrm{el}(W_0)$ has component-terminal objects (i.e. W_0 is multi-representable). Call these objects **generic**.

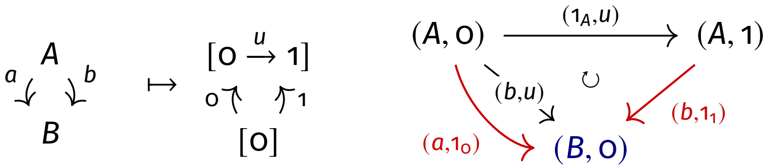
When is a weight $W : \mathcal{A}^{\mathsf{op}} \to \mathbf{Cat}$ oplax?

Recall: A weight W is *PIE* precisely when $\mathsf{el}(W_0)$ has component-terminal objects (i.e. W_0 is multi-representable). Call these objects **generic**.

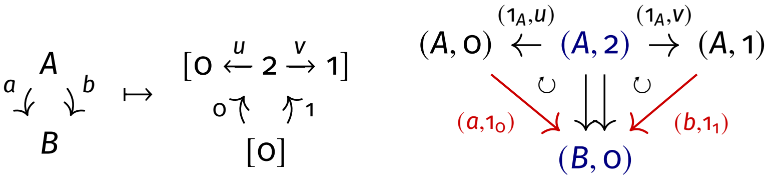
Thm. W is oplax if it is PIE and $\mathsf{el}(W)$ additionally satisfies:

For any $f : y \to x$ into a generic and a chosen-cartesian $g : y \to z$, the hom-category from g to f in the lax coslice $(y \Downarrow \mathsf{el}(W))$ has a single connected component.

Coinserters



"Span coinserters"



Further results

Subclasses of oplax weights:

- conical oplax colimits of *oplax* (or *normal oplax*) functors from 1-categories or groupoids: presaturated and admits a nice recognition result. Free cocompletions given by a 2-categorical "Fam" construction.
- coKleisli weights (conical from $\mathbb{B}\Delta_+^{\mathsf{op}}$): presaturated, cocompletion given in (Lack and Street 2002).

The class of conical oplax colimits of pseudo or strict functors from 1-categories is *not* presaturated.

Further questions

What are the oplax versions of *(semi)-flexible* weights?

Is there a finite class of weights which generates all oplax weights, as for PIE weights?

Is there a similar characterisation of weights for *pseudo*-colimits?

Thanks!

References

- Lack, Stephen (2005). "Limits for lax morphisms". In: *Appl. Categ. Structures* 13.3, pp. 189–203. ISSN: 0927-2852. DOI: 10.1007/s10485-005-2958-5. URL: https://doi.org/10.1007/s10485-005-2958-5.
- Lack, Stephen and Michael Shulman (2012). "Enhanced 2-categories and limits for lax morphisms". In: *Adv. Math.* 229.1, pp. 294–356. ISSN: 0001-8708. DOI: 10.1016/j.aim.2011.08.014. URL: https://doi.org/10.1016/j.aim.2011.08.014.
- Lack, Stephen and Ross Street (2002). "The formal theory of monads. II". In: vol. 175. 1-3. Special volume celebrating the 70th birthday of Professor Max Kelly, pp. 243–265. DOI: 10.1016/S0022-4049(02)00137-8. URL: https://doi.org/10.1016/S0022-4049(02)00137-8.
- Lambert, Michael (2024). "Discrete 2-fibrations". In: *High. Struct.* 8.1, pp. 54–96.
- Power, John and Edmund Robinson (1991). "A characterization of pie limits". In: *Math. Proc. Cambridge Philos. Soc.* 110.1, pp. 33–47. ISSN: 0305-0041. DOI: 10.1017/S0305004100070092. URL: https://doi.org/10.1017/S0305004100070092.
- Street, Ross (1976). "Limits indexed by category-valued **2**-functors". In: *J. Pure Appl. Algebra* 8.2, pp. 149–181. ISSN: 0022-4049. DOI: 10.1016/0022-4049(76)90013-X. URL: https://doi.org/10.1016/0022-4049(76)90013-X.
- Street, Ross and R. F. C. Walters (1973). "The comprehensive factorization of a functor". In: *Bull. Amer. Math. Soc.* 79, pp. 936–941. ISSN: 0002-9904. DOI: 10.1090/S0002-9904-1973-13268-9. URL: https://doi.org/10.1090/S0002-9904-1973-13268-9.