Building pretorsion theories from torsion theories

Federico Campanini

joint work in progress with Francesca Fedele and Emine Yıldırım

Building (some) pretorsion theories from torsion theories and lattices of pretorsion classes

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Torsion theories in abelian categories [Dickson]

Definition: Let $\mathbb{C}$ be an abelian category.
A pair $(\mathcal{T}, \mathcal{F})$ of full replete subcategories of $\mathbb{C}$ is a torsion theory if

- $\mathrm{Hom}(T, F) = 0$ for all $T \in \mathcal{T}$, $F \in \mathcal{F}$;
- for every $X \in \mathbb{C}$ there exists a short exact sequence

$$0 \to T_X \to X \to F_X \to 0 \qquad \text{with } T \in \mathcal{T}, F \in \mathcal{F}.$$

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Example:
$(\mathcal{T}, \mathcal{F})$ in the category Ab of abelian groups, where

- $\mathcal{T}$ = torsion groups;
- $\mathcal{F}$ = torsionfree groups

$$0 \longrightarrow t(G) \longrightarrow G \longrightarrow G/t(G) \longrightarrow 0 \qquad s.e.s$$

with $t(G)$ = torsion subgroup of G.

Torsion theories in pointed categories

[Many authors in several contexts]

Definition: Let $\mathbb{C}$ be any pointed category.
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Definition: Let $\mathbb{C}$ be any category.
A pair $(\mathcal{T}, \mathcal{F})$ of full replete subcategories of $\mathbb{C}$ is a pretorsion theory if

- $\mathrm{Hom}(T, F) = \mathrm{Triv}(T, F)$ for all $T \in \mathcal{T},\ F \in \mathcal{F}$;
- for every $X \in \mathbb{C}$ there exists a short $\mathcal{Z}$-exact sequence

$$T_X \to X \to F_X \qquad \text{with } T \in \mathcal{T}, F \in \mathcal{F}.$$

Comparable torsion theories [—, Fedele]:

Let $\mathbb{C}$ be a pointed category and consider two torsion theories $(\mathcal{T}_1, \mathcal{F}_1)$ and $(\mathcal{T}_2, \mathcal{F}_2)$ in it.

The following conditions are equivalent:

(i) $\mathcal{T}_2 \subseteq \mathcal{T}_1$ $(\mathcal{F}_1 \subseteq \mathcal{F}_2)$

(ii) $(\mathcal{T}_1, \mathcal{F}_2)$ is a pretorsion theory.

Moreover, if these conditions hold, then the $\mathcal{Z}$-short exact sequence of an object $X \in \mathbb{C}$ is given by

$$T_1 X \longrightarrow X \longrightarrow F_2 X$$

Notice: no hypothesis are required for $\mathbb{C}$ or the torsion theories.

Some remarks:

- If $\mathcal{T}$ is a torsion class, then $\mathcal{F}$ is uniquely determined

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- Pretorsion classes in $\mathbb{C}$ are precisely the monocoreflective subcategories of $\mathbb{C}$.

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- All the important information can be encoded into its Aulander-Reiten quiver.
- Torsion and pretorsion classes are quite easy to detect.

Example

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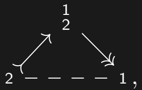

Let me try to draw a picture...

Classification of distributive lattices for finite representation type (here $Q = A_n, D_n, E_6, E_7, E_8$)

The poset of pretorsion classes is a complete lattice, with meet and join given, for every $\mathcal{T}_1$ and $\mathcal{T}_2$, by

$$\mathcal{T}_1 \wedge \mathcal{T}_2 = \mathcal{T}_1 \cap \mathcal{T}_2 \qquad \text{and} \qquad \mathcal{T}_1 \vee \mathcal{T}_2 = \langle \mathcal{T}_1 \cup \mathcal{T}_2 \rangle_t .$$

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Result 1 [— , Fedele, Yıldırım]

The lattice of pretorsion classes is distributive if and only if $\mathrm{add}\{\mathcal{T}_1 \cup \mathcal{T}_2\} = \langle \mathcal{T}_1 \cup \mathcal{T}_2 \rangle_t$ for every pair of pretorsion classes $\mathcal{T}_1$ and $\mathcal{T}_2$ in $\mathrm{mod}\,kQ$.

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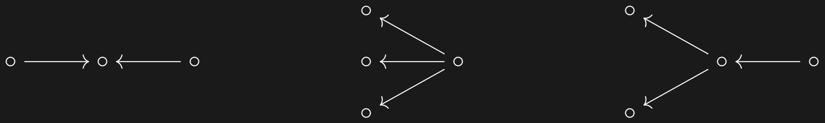
Result 2 [— , Fedele, Yıldırım]

The lattice of pretorsion classes is distributive if and only if Q does not contain subquivers of the form



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Result 3 [— , Fedele, Yıldırım]

There is a bijection between the isomorphism classes of indecomposable modules and the join-irreducible elements of the lattice of pretorsion classes, given by $M \mapsto \langle M \rangle_t$. Moreover, the join-irreducible elements are torsion classes.

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Result 4 [— , Fedele, Yıldırım]

If the lattice of pretorsion classes is distributive, than it is the distributive completion of the lattice of torsion classes.

Thank you